ROBUST DIRECTION FINDING IN THE FLUCTUATING OCEANS BY COMPLEX L1-NORM PRINCIPAL COMPONENT ANALYSIS

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Abstract: *We consider the problem of robust direction-of-arrival (DoA) estimation in dynamic ocean environments. Robust direction finding of underwater acoustic signals transmitted from known locations may enable accurate localization of a receiver node underwater, as well as enhance link communication rate by instructing the receiver to listen for transmissions from a specific direction. We propose to estimate the DoA of underwater acoustic signals via subspace methods, executed at a receiver array, that involve performing what is known as principal-component analysis (PCA) for finding the L2-norm principal vector subspace of the recorded signal snapshots. However, in practice coherence loss, which typically arises from dynamic wavefront fluctuations due to internal waves, scattering from the sea surface and/or bottom and other unknown environmental parameters, may result in recorded signal snapshots that may be corrupted by faulty measurements, leading to an inaccurate estimation of the DoA and source position. We propose to model the loss of coherence as multiplicative random noise applied to the measured acoustic signal. In such cases, L2-norm PCA methods suffer from significant performance degradation. Motivated by the resistance of novel L1-norm-derived subspaces against the impact of irregular, highly deviating points in reduced-dimensionality data approximations, we propose to employ L1-norm (absolute error) maximum-projection PCA of the antenna array measurements and evaluate the performance of a novel, outlier-resistant DoA estimation algorithm. Experimental assessment of the proposed DoA algorithm is conducted over underwater measurements acquired from an ultrasonic transmitter source (at 2.25MHz) operating in a controlled water tank environment that reproduces the effect of a fluctuating ocean. This experimental configuration allows to accurately reproduce the effects of spatial medium fluctuations on the propagated sound waves, in a fully reproducible and monitored fashion. The benchmark of the proposed DoA estimation algorithm on this experimental dataset represents a preliminary step before validating its performance on at-sea recorded data.*

Keywords: *Undersea localization, robust direction finding, complex L1-PCA*

1. INTRODUCTION

Fundamental to the success of underwater communications is robust localization and tracking to support position-aware data routing and autonomous vehicle navigation in the GPS-less oceanic environment. GPS-free localization schemes proposed for terrestrial radio networks involve intensive message exchanges, and therefore are not suitable for the low-bandwidth, high-latency undersea acoustic channel. For this reason, existing state-of-art approaches for undersea localization and tracking either consider expensive inertial sensors [1], geophysical-based [2], or acoustic communication techniques [3]. Inertial/dead reckoning techniques exhibit position error growth that is unbounded with traveled distance due to inertial sensor drifts. High cost and power consumption of quality inertial measurement sensors precludes their use in small, low-cost undersea vehicles. Geophysical-based techniques exhibit bounded position error, but they are challenged by the visual and acoustic range of cameras and sonar, that compromise accurate detection, identification, and classification of features extracted from corresponding imaging information. Acoustic-based localization techniques rely on angle or distance measurements between communicating nodes that are usually collected by received-signal-strength indicator (RSSI), time-of-arrival (ToA), time of flight (ToF), time-difference-of-arrival (TDoA), and angle-of-arrival (AoA) [4] methods. However, long propagation delays and the multipath nature of the undersea environment, especially in shallow water, results in significant errors and outliers in acoustic-based measurements. In this context, acoustic beacon ranging proposals in the literature have been seen to provide unreliable location estimates due to stringent time synchronization requirements and variable path loss. Similarly, acoustic localization based on multiple-signal-classification (MUSIC)-type and maximum-likelihood (ML) DoA estimation methods have their own limitations. MUSIC-type schemes are more efficient in terms of hardware and execution time requirements compared to their ML counterparts. However, they are limited in the case of loss of coherence (or correlation) due to ocean fluctuations such as internal waves or in the presence of multipath interference due to shallow water.

Conventional subspace-based DoA-estimation methods, such as the celebrated MUSIC [5], seek to exploit the inherent separability of the signal and noise subspaces. Therefore, they rely largely on the eigenstructure of the (estimated) received-signal correlation matrix, or the L2-norm-principal subspaces of the recorded snapshots, obtained usually by means of familiar singular-value-decomposition (SVD) [6] of the received data matrix. In unobstructed (normal) system operation, such methods are well known to afford high target-angle resolution, while, in Gaussian noise, offer unbiased and asymptotically consistent DoA estimates [7].

However, the coherence loss, which typically arises from dynamic ocean fluctuations and unknown environmental parameters may take the form of multiplicative colored random noise that is applied to the recorded measurements. In such cases, the performance of L2-subspace-based DoA estimation methods may be significantly degraded, due to corrupted signal snapshots that are represented in the receiver data matrix by points that lie far off the nominal-signal subspace. Squared-fitting-error minimizers [8] such as L2-principal subspaces are highly responsive to such outlier data points, thus L2-subspace-based DoA estimators are inevitably misled.

On the other hand, absolute-error minimizers place less emphasis on individual data-point divergence. More specifically, L1-PCA of real-valued data matrices has attracted extensive documented research since the mid-twentieth century [9] and even more so in the past decade. Work in [10] focused on L1-PCA in the form of residual error minimization. However, minimum error L1-PCA has to date no known solution for its general multiple component case. Work in [11] considered projection maximization L1-PCA where data projection is measured by means of the L1-norm, while the first exact solvers for L1-PCA of real valued data are provided in [12], [13]. Simulation studies highlight the sturdy resistance of L1-derived subspaces against outlier-inflicted data corruption. Applications of the proposed algorithms include dimensionality reduction, image conditioning/restoration and direction finding. Lp-norm derived subspaces have been employed in [14] for robust MUSIC-like DoA estimation in impulsive noise environments. Recent work in [15] established theoretical and algorithmic foundations of L1-PCA for complex-valued data matrices.

In this paper, motivated by the resistance of L1-norm derived subspaces against outliers and/or impulsive noise data corruption, we propose to employ L1-norm (absolute-error) maximum-projection PCA on complex data measurements [15] from a hydrophone array and evaluate an outlier-resistant MUSIC-like DoA estimation algorithm on experimental data from a scaled water tank experiment [14]. The tank experiment aims to reproduce the effect of ocean fluctuations on the received pressure field. The coherence loss is mimicked by means of a random lens placed between an ultrasonic transmitter source (at 2.25 MHz) and the array of hydrophones. We assess the direction finding performance of complex L1-norm PCA algorithms [15] and compare to conventional MUSIC (L2) estimation methods by means of the standard root-mean square error (RMSE).

2. SYSTEM MODEL

We consider an underwater uniform linear array (ULA) of M elements. The length-M array response vector for a source impinging on the antenna array at an arbitrary direction of arrival $\theta \epsilon (-\pi/2, \pi/2]$ is given by:

$$\mathbf{s}(\theta) \triangleq \left[1, e^{-j\frac{2\pi f_c d \sin(\theta)}{c}}, \ldots, e^{-j\frac{(D-1)2\pi f_c d \sin(\theta)}{c}}\right]^T \in \mathbb{C}^{M\times 1} \tag{1}$$

where $(\cdot)^T$ denotes the transpose operator, f_c is the carrier frequency, c is the signal propagation speed underwater, and d is the fixed inter-element spacing of the array. Let θ be the direction of arrival of a monochromatic source that emits an acoustic signal at frequency f_c. The acoustic pressure field received at the antenna array is represented by its Discrete Fourier Transform (DFT). Thus, the n-th observation vector, for $n = 1, 2, \cdots, N$ is of the form:

$$\mathbf{y}_n = x_n(\theta)\mathbf{s}(\theta) + \mathbf{n}_n \in \mathbb{C}^{M\times 1} \tag{2}$$

where $x_n(\theta) \in \mathbb{C}$ denotes the n-th snapshot of the source signal and $\boldsymbol{n}_n \in \mathbb{C}^M$ accounts for the additive zero-mean and white circularly-symmetric complex Gaussian noise vector at the n-th snapshot, with per-element variance σ^2, that is $\boldsymbol{n}_n \sim CN\ (0, \sigma^2 \boldsymbol{I}_M)$, where $\boldsymbol{I}_M$ is the identity matrix.

In order to model the loss of coherence along the antenna array, we introduce a random variable $\boldsymbol{\psi}_n \in \mathbb{C}^M$ such that:

$$\mathbf{y}_n = x_n(\theta)\mathbf{s}(\theta) \otimes \psi_n + \mathbf{n}_n \in \mathbb{C}^{M\times 1} \tag{3}$$

where $\otimes$ denotes the hadamard product operator and $\boldsymbol{\psi}_n$ is assumed to account for the multiplicative zero-mean and complex normal distributed phase noise vector at the n-th snapshot with covariance matrix $\boldsymbol{\Sigma}_n$. Work in [18] has considered a similar model to statistically characterize particular regimes of fluctuation. In fact, such multiplicative complex noise is well-suited to model as additive phase noise. Considering the model in Eq. (3), we then assume that the loss of coherence is due to a perturbation of the phase wavefront. In order to facilitate the experimental setup, the covariance matrix $\boldsymbol{\Sigma}_n$ is assumed to remain constant in time, i.e., we consider only spatial fluctuations of the propagating medium.

3. SUBSPACE BASED DIRECTION OF ARRIVAL ESTIMATION

In regular system operation (i.e., when there are no spatial fluctuations) the one-dimensional principal subspace of the observed snapshots at the antenna array is the span of the steering vector $\boldsymbol{s}(\theta)$. Perfect knowledge of $\boldsymbol{s}(\theta)$ can lead to perfect knowledge of θ. In other words, for any $\theta \epsilon (-\pi/2, \pi/2]$, $M - |\, \boldsymbol{s}^{\boldsymbol{H}}(\varphi)\boldsymbol{s}(\theta)| = 0$, if and only if $\theta = \varphi$. Unfortunately, perfect knowledge of $\boldsymbol{s}(\theta)$ is not possible with finite number of observations $\{\boldsymbol{y}_n\}_{n=1}^{N}$; $\boldsymbol{s}(\theta)$ can only be estimated from the received snapshots. In particular, the autocorrelation matrix of the complex valued observations is:

$$\mathbf{R}_y = \mathrm{E}\{\mathbf{y}_\mathrm{n}\mathbf{y}_\mathrm{n}^\mathrm{H}\} = \mathbf{s}(\theta)\mathbf{R}_\mathrm{x}\mathbf{s}^\mathrm{H}(\theta) + \sigma^2\mathrm{I}_\mathrm{M} \tag{4}$$

where $\boldsymbol{R}_x$ is the autocorrelation matrix of the source signal. Given the eigenvalue-decomposition (EVD) of $\boldsymbol{R}_y$:

$$\mathbf{R}_y \overset{\mathrm{evd}}{=} \mathbf{U}\left(\boldsymbol{\Lambda} + \sigma^2\mathbf{I}_M\right)\mathbf{U}^H \tag{5}$$

where $\boldsymbol{\Lambda} = \boldsymbol{diag}[\lambda_1 \quad \cdots \quad \lambda_M]$ is the diagonal matrix containing the sorted eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$ and $\boldsymbol{U}$ is the corresponding $M{\times}M$ orthonormal eigenmatrix, the leftmost column of $\boldsymbol{U}_{:,\mathbf{1}}$ constitutes a complete orthonormal basis for the source signal subspace. Then the solution $\boldsymbol{q}_{opt} \in \mathbb{C}^M$ to the L2-principal subspace problem:

$$\underset{\mathbf{q}\in\mathbb{C}^M;\ \|\mathbf{q}\|_2=1}{\mathrm{argmax}} \quad \mathrm{Tr}\left(\mathbf{q}^H\mathbf{R}_y\mathbf{q}\right) \tag{6}$$

coincides with $\boldsymbol{U}_{:,\mathbf{1}}$ and offers a complete characterization of the signal subspace. As a result, any $\theta \epsilon (-\pi/2, \pi/2]$ can be optimally examined for source signal presence by:

$$\theta \in (-\pi/2, \pi/2] \Leftrightarrow (\mathbf{I}_M - \mathbf{q}_\mathrm{opt}\mathbf{q}_\mathrm{opt}^H)\mathbf{s}(\theta) = \mathbf{0}_M \tag{7}$$

In practice, perfect knowledge of $\boldsymbol{R}_y$ cannot be assumed, instead $\boldsymbol{R}_y$ may be sample average estimated over the N complex-valued snapshot vectors as $\boldsymbol{R}_y \triangleq \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{y_n}\,\boldsymbol{y}_n^H$.

The maximization argument in Eq. (6) is accordingly calculated by $\frac{1}{N}Tr(\boldsymbol{q}^H\boldsymbol{Y}\boldsymbol{Y}^H\boldsymbol{q}) = \frac{1}{N}\|\boldsymbol{q}^H\boldsymbol{Y}\|_2^2$. Consequently, the solution to the problem in Eq. (6) is equivalent to the estimation of the L2 –principal subspace from a number of samples in $\mathbb{C}^M$ that is formulated as:

$$\mathbf{q}_{\mathrm{L2}} = \underset{\mathbf{q}\in\mathbb{C}^M;\ \|\mathbf{q}\|_2=1}{\mathrm{argmax}} \quad \|\mathbf{q}^H\mathbf{Y}\|_2 \tag{8}$$

where $\boldsymbol{Y} \triangleq [\boldsymbol{y}_1 \quad \cdots \quad \boldsymbol{y}_N] \in \mathbb{C}^{M\times N}$ and $\|\cdot\|_\mathbf{2}$ denotes the L2-norm. The solution to Eq. (4) $\boldsymbol{q}_{L2}$ is the left singular vector of the complex-valued matrix $\boldsymbol{Y}$ that corresponds to its highest singular value. For fixed number of observation snapshots N and additive white Gaussian noise data corruption as described in Eq. (2), $\boldsymbol{q}_{L2}(N)$ is the maximum likelihood (ML) estimate of the true signal subspace $\boldsymbol{q}_{true} = \frac{\boldsymbol{s}(\theta)}{\sqrt{M}}$. As N increases to infinity, $\|\boldsymbol{q}_{L2}(N) - \boldsymbol{q}_{true}\|$ converges to zero in probability for every norm. In view of Eq. (7), the direction of arrival for the source signal is estimated as the argument that corresponds to the highest peak of the power spectrum:

$$P_{L2}(\theta) = \|(\mathbf{I}_M - \mathbf{q}_{L2}\mathbf{q}_{L2}^H)\mathbf{s}(\theta)\|_2^{-1} \tag{9}$$

which is proved to be equivalent to the familiar MUltiple SIgnal Classification procedure (MUSIC) [5].

4. ROBUST DIRECTION FINDING WITH COMPLEX L1-PCA

We now steer our focus to the scenario of interest, where each snapshot is corrupted by multiplicative phase noise that emulates the loss of coherence in the receiver hydrophone array. Typically, the closer the antenna elements, the more correlated the acoustic pressure. On the other hand, two antenna elements that are far from each other may capture different signal perturbations. Consequently, the received signals should be less correlated. Considering that the coherence loss can be modelled as multiplicative random noise applied to the measured acoustic signal, then signal and noise subspaces are not separable by means of EVD operation any more. Another interesting observation is that, in a contaminated data record, the corrupted snapshots are expected to lie far away from the signal subspace snapshots. Importantly, as practitioners have long observed, squared-value expressions, such as L2-norm, put significant emphasis on extreme errors and render accordingly principal subspaces that are particularly sensitive to outlier-inflicted data corruption.

Recent studies in the field of L1-principal component analysis for complex-valued data [15] have demonstrated that L1-principal subspaces are far more resistant to sporadic contamination of data by outliers than L2-principal subspaces. Motivated by these observations, we propose to examine the direction finding performance of a DoA estimation algorithm that instead of $span(\boldsymbol{q}_{L2})$ uses the one-dimensional L1-principal subspace of the received complex-valued data matrix, defined by the span of :

$$\mathbf{q}_{\mathrm{L1}} = \underset{\mathbf{q}\in\mathbb{C}^M;\ \|\mathbf{q}\|_2=1}{\operatorname{argmax}} \|\mathbf{q}^H\mathbf{Y}\|_1 \tag{10}$$

where $\|\cdot\|_1$ denotes the L1-norm. Thereafter, the direction of arrival of the source signal of interest will be calculated in accordance to Eq. (9) as the argument that corresponds to the highest peak of the L1 principal subspace power spectrum that is defined as:

$$P_{\mathrm{L1}}(\theta) = \| \left(\mathbf{I}_M - \mathbf{q}_{L1}\mathbf{q}_{L1}^H\right) \mathbf{s}(\theta)\|_2^{-1} \tag{11}$$

By not placing squared emphasis on the magnitude of each point (as L2-PCA does) L1-PCA is far more resistant to outlying, peripheral data points. Recent studies have shown that when the processed data are not outlier corrupted the solutions of L1-PCA and L2-PCA describe a nearly identical subspace. Although L1-principal subspace finding in the form of Eq. (10) is not a new problem in the literature, the first exact/optimal solvers for real-valued data were provided in [12], [13]. Specifically, [13] proved that real-valued L1-principal component analysis can be converted into a combinatorial problem over antipodal binary variables (±1) solvable with intrinsic complexity, polynomial in the data record size $\mathcal{O}(N^{\boldsymbol{rank}(Y)})$. In contrast to real-valued L1-PCA, complex L1-PCA –in the form of Eq. (10)- has no obvious connection to a combinatorial problem. Until recently, no finite-step algorithm (exponential or other) was known for providing an optimal solution to Eq. (10). Work in [15] has proved that Eq. (10) can be casted as an optimization problem over the set of unimodular vectors and provided two fast algorithms to solve Eq. (10) optimally.

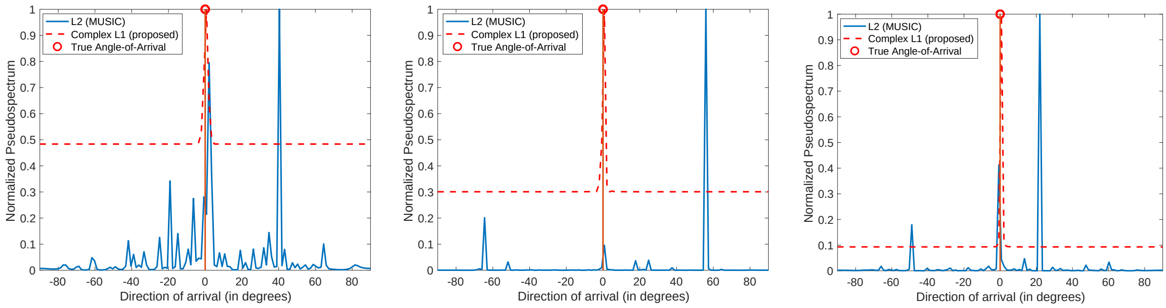

Fig.1: DoA estimation spectra P_{L1} and P_{L2} for unsaturated [US1, 17], partially saturated [PS1, 17], and fully saturated [FS1, 17] fluctuation categories in the experimental configuration (from left to right).

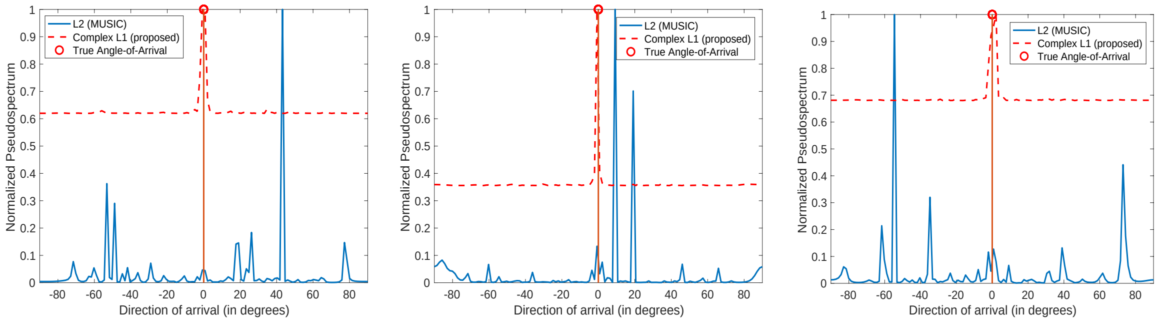

Fig.2: DoA estimation spectra P_{L1} and P_{L2} for unsaturated [US1, 17], partially saturated [PS1, 17], and fully saturated [FS1, 17] fluctuation categories in the P3DCOM configuration (from left to right).

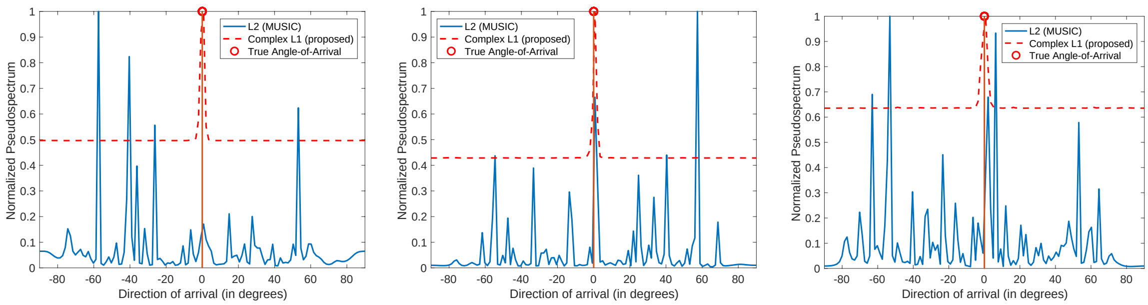

Fig.3: DoA estimation spectra P_{L1} and P_{L2} for unsaturated [US1, 17], partially saturated [PS1, 17], and fully saturated [FS1, 17] fluctuation categories in the P3DTEx configuration (from left to right).

5. EXPERIMENTAL RESULTS

We leverage direct complex L1-PCA introduced in [Algorithm 1, 15] to experimentally assess the direction finding performance of both L2-PCA and L1-PCA by carrying out DoA estimation over complex-valued measurements acquired from an ultrasonic transmitter source that operates in a controlled water tank environment. In particular, a moving hydrophone simulates a virtual array of hydrophones and measures the monochromatic continuous acoustic signal that is emitted from a fixed transducer [17]. We consider a CW chirp signal that is transmitted at f_c = 2.25 MHz with duration 22.2 μs, amplitude of 5 V, and wavelength λ = 0.658 mm. The signal is transmitted through a random faced acoustic lens that is used to to induce a spatially fluctuating sound pressure field [17]. Measurements are conducted in a 3m long, 1m wide and 1m deep, water tank filled with fresh water. The temperature was controlled by a probe continuously. The equipment was located in the middle of the tank depth and it was possible to ignore the influence of bottom and surface reflections. Automatic

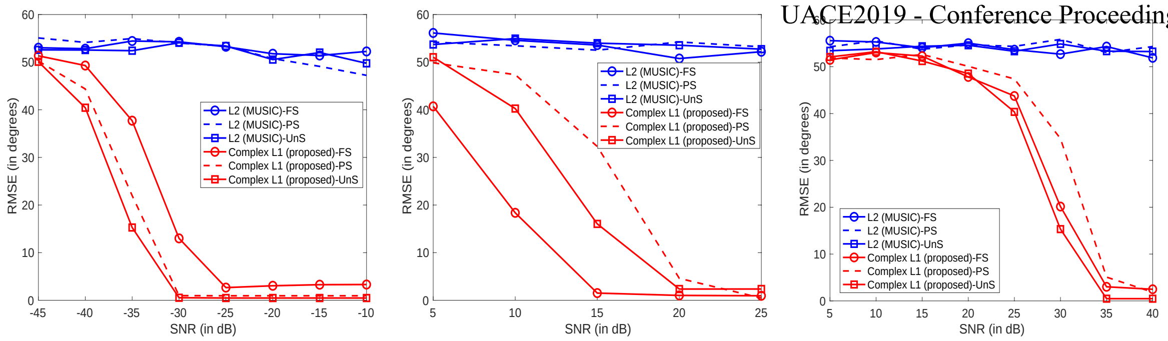

Fig.4: RMSE versus SNR for unsaturated [US1, 17], partially saturated [PS1, 17], and fully saturated [FS1, 17] fluctuation categories for experimental, PD3COM and PD3TEX configurations (from left to right).

displacements of the hydrophone on a vertical axis were used to simulate vertical arrays of $M = 64$ hydrophones, separated by a distance of 0.3 mm. More details and a diagram of the experimental setup can be found in [17].

Signal distortions from the random lens that is introduced between the transducer and the receiver array are classified into three categories: (i) unsaturated (UnS) i.e., the only observed perturbation arises from a single fluctuating path; (ii) partially saturated (PS) i.e., multipath correlation occur as well; (iii) fully saturated (FS) i.e., each multipath signal is subject to its own environmental fluctuation. Automatic displacements of the source and the receiver allowed us to acquire several realizations/snapshots of the same fluctuation category.

To obtain relevant comparisons between the configuration studied in a water tank and real scale oceanic cases, we use measurements from a dimensional analysis based on P3DTEx (for propagation in a 3D tank experiment configuration) and P3DCOM (for propagation in a 3D corresponding oceanic medium) software; both software tools are based on a 3D parabolic model). Oceanic configurations used in these models correspond to the propagation of acoustic waves in the mid-frequency band (1-15 kHz) over distances of the order 1-10 km. They correspond to short times of propagation compared with the daily period of medium fluctuations so that the studied phenomena are considered spatially but frozen in time.

Fig. 1 depicts the DoA estimation spectra P_{L1} and P_{L2} using L1-PCA [15] and L2-PCA methods, respectively, to estimate the angular position of the ultrasound source from experimental water tank measurements. Fig. 2 and Fig. 3 demonstrate the robustness of direct complex L1-CA versus conventional MUSIC (L2-PCA) method for DoA estimation using complex-valued measurements from P3DCOM and P3DTEx configurations, respectively. Parameters for the water tank experiment, the scaled tank experiment as well as the corresponding ocean configuration can be found in [17]. For the experiments above we consider a constant direction of arrival at $\theta = 0°$. Furthermore, we consider evaluation of the DoA spectra with $M = 64$ hydrophones and $N = 20$ snapshots for all the above configurations. The variance of the additive white noise for the experimental results reported in Fig. 1 is controlled by setting link signal-to-noise ratio (SNR) equal to -15dB. The link SNR for scaled tank and oceanic configurations using P3DCOM (Fig. 2) and P3DTEx (Fig. 3) software tools is fixed at 15dB.

Fig. 4 evaluates the average DoA estimation performance of the L2-PCA-based and L1-PCA-based methods for different SNR values by means of RMSE defined as $RMSE \triangleq \sqrt{\frac{1}{1000}\sum_{m=1}^{1000}|\theta - \hat{\theta}(m)|^2}$, where $\hat{\theta}(m)$ is the estimate of the true angle of arrival θ in the m-th experiment. At each iteration, we randomly select a DoA, and sample $N = 20$ corresponding snapshots (considering a stationary process scenario). The superiority of the proposed L1-PCA method over the conventional MUSIC (L2) method is demonstrated across a wide range of SNRs for all three fluctuation categories.

It is evident that the proposed direction finding method offers significant DoA estimation accuracy even when the recorded snapshots at the hydrophone array may be corrupted by outlying entries due to the dynamic spatial fluctuation of the propagation medium.

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