Modeling sonar performance using J-divergence

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Abstract: *Detection performance in sonar systems is typically quantified by the probabilities of detection (P_d) and false alarm (P_f) in a single sonar resolution cell. When system-level performance is paramount, however, the inherent inaccuracies in mapping to the cell-level (P_d, P_f) detector operating point encourage consideration of approximate performance measures such as J-divergence. The basic application of J-divergence to modeling sonar detection performance is presented in this paper. The properties of J-divergence making it an appealing choice are covered: a single scalar measure of detection performance (i.e., a detection currency), summing to accrue J-divergence across multiple independent measurements (e.g., from multiple source signals, waveforms, or arrays), a data-processing inequality dictating that processing cannot improve J-divergence, and an asymptotic relationship to the multiple-measurement receiver operating characteristic curve. Simple forward models of J-divergence are presented for matched filters and energy detectors when applied to several standard signal models in Gaussian noise. Similarly accessible results for inverse modeling of the design signal-to-noise power ratio (termed "*DJ*") required to achieve a specified level of J-divergence detection currency are presented. This provides a direct replacement for the detection threshold (*DT*) term in the sonar equation that is easier to evaluate and apply. The efficacy of the approach is demonstrated by comparing* DJ *and* DT *for matched filters and energy detectors.*

Keywords: *detection performance, J-divergence, detection threshold*

1. INTRODUCTION

Sonar performance models are used when designing or tuning sonar systems, evaluating their performance in specific environments, or when comparing competing systems or designs. Modeling the detection performance of a sonar system often entails (i) the basic sonar equation to obtain the signal-to-noise power ratio (SNR) after coherent detection processing, (ii) forward modeling of detection performance given the SNR, and (iii) inverse modeling of the *design* SNR required to achieve a detection-performance specification. Traditional approaches (e.g., [1, Ch. 12]) define detection performance using the probabilities of detection and false alarm (P_d, P_f) observed for a single sonar resolution cell. However, many systems utilize measurements obtained from multiple waveforms, frequency bands, sensors, or platforms. Extending the traditional analysis to multiple-measurement detection probabilities $(\mathrm{PDM}, \mathrm{PFM})$ is often so complicated that only simple scenarios are considered where single-measurement tools can be applied to approximate multiple-measurement detection performance. An alternative approach based on J-divergence [2,3] is proposed here for multiple-measurement systems.

Although the traditional approach provides an accurate representation of cell-level detection performance, it often inaccurately represents system-level performance (e.g., by approximating the multiple-measurement detector or not precisely converting P_f to a false-alarm rate). These inaccuracies between cell- and system-level modeling encourage considering other metrics that may be approximate at the cell level, but easier to apply to multiple measurements. Potentially suitable alternatives include the total SNR, detection index, and J-divergence. Of these, the J-divergence provides a useful balance across considerations of accuracy, evaluation difficulty, applicability throughout the signal and information processing chain (SIPC), and extension to more complicated scenarios. In contrast to the (P_d, P_f) operating point, J-divergence distills the receiver operating characteristic (ROC) curve into a *detection currency* without a specific operating point.

The basic tools required to model sonar detection performance with J-divergence are presented in this paper. A brief background on J-divergence and its properties are presented in Sect. 2. An asymptotic relationship to the ROC curve is used to characterize low, medium, and high levels of quality for J-divergence detection currency. With a focus on matched filters and energy detectors, the application to sonar found in Sect. 3 presents forward models for J-divergence as a function of the SNR when detecting the basic sonar signal models (Gaussian-fluctuating, deterministic, and Rician) in Gaussian noise. These functional relationships are then inverted to obtain the design SNR (termed “DJ”) as a function of a detection-currency specification, yielding a direct replacement for the detection threshold (DT) term in the sonar equation. Although the Gaussian-fluctuating signal permits analytic solutions, approximations are developed for the others, making them similarly convenient to apply. Finally, an example comparison is presented in Sect. 4 between DT and DJ for the aforementioned performance levels.

2. J-DIVERGENCE AND ITS PROPERTIES

The J-divergence can be found in the early work of Jeffreys [2], [3, Sect. 3.10, eq. 1]. It is also a member of the Ali-Silvey [4] class of distance measures between two statistical distributions. Suppose T is the detector decision statistic and $f_0(t)$ and $f_1(t)$ are its probability density functions (PDFs) under the noise-only (H_0) and signal-present (H_1) hypotheses, respectively.

The J-divergence between these distributions is

$$J = \int_{-\infty}^{\infty} [f_1(t) - f_0(t)] \log\left[\frac{f_1(t)}{f_0(t)}\right] \, \mathrm{d}t, \tag{1}$$

which is equivalent to the difference in the average log-likelihood ratio (LLR) between the two hypotheses. Recalling the optimality of the LLR (maximizing P_d for a given P_f), it is clear that an increase in J-divergence is indicative of an improvement in detection performance.

The key properties of J-divergence supporting its use in modeling sonar detection performance are (i) it sums over independent measurements, (ii) it can be evaluated throughout the SIPC, and (iii) it has an asymptotic (large sample/low SNR) relationship to the multiple-measurement operating point. The first property means accounting for multiple dissimilar measurements is straightforward and only requires evaluating the J-divergence on each individual measurement. If J_m is the J-divergence for the mth of M independent measurements, the total J-divergence is simply

$$J = \sum_{m=1}^{M} J_m. \tag{2}$$

The level of difficulty for evaluating multiple measurements is essentially the same as it is for a single measurement, which is not the case when evaluating $(\mathrm{PDM}, \mathrm{PFM})$. Because J-divergence represents the potential for detection performance over multiple measurements, the sum in (2) inherently assumes the measurements are combined optimally. The applicability of the data processing inequality (i.e., processing can only maintain or decrease J-divergence) implies (2) can be used as an upper bound on performance, which circumvents the need to define and evaluate the multiple-measurement detector. It also permits evaluation at different points in the SIPC, with increasing prediction accuracy as more processing steps are included.

The final key property relates J-divergence to the asymptotic ROC curve of the optimal multiple-measurement detector according to

$$J = \left[\Phi^{-1}(1 - \mathrm{PFM}) - \Phi^{-1}(1 - \mathrm{PDM})\right]^2, \tag{3}$$

where $\Phi^{-1}(p)$ is the functional inverse of the standard-normal cumulative distribution function (CDF). This result implies that J-divergence will be an exact surrogate for detection performance asymptotically as the number of (increasingly weak) measurements tends to infinity. As will be seen, the relationship holds accurately enough when the number of measurements is above about five for J-divergence to be useful in representing detection operating points.

2.1. CONVERSION TO DECIBELS AND NOMINAL OPERATING POINTS

As might be expected of a detection currency, J-divergence is monotonically related to SNR and consequently subject to a similarly large span of values. It will therefore be convenient to describe it using a decibel notation. Based on the example to be presented in Sect. 3, an argument can be made for converting J-divergence to decibels via

$$\mathrm{J_{dB}} = 5 \log_{10} J = 10 \log_{10} \sqrt{J} \qquad \text{[units: dB]}. \tag{4}$$

owing to its low-SNR relationship to a squared ratio of intensities. The decibel quantity in (4) will be referred to as J-divergence *detection currency*.

In order for J-divergence to be of practical use, the scale of detection currency needs to be interpreted. The asymptotic relationship in (3) provides a potentially coarse link between a detection operating point and detection currency. Three different levels of quality in detector operating points are shown in Table 1 along with the corresponding asymptotic-ROC-related detection currency ($\mathrm{J}_{\mathrm{dB}}^{\mathrm{asy}}$). However, in many scenarios this relationship does not need to be precise and the nominal levels of 5, 8, and 10 dB of detection currency ($\mathrm{J}_{\mathrm{dB}}^{\mathrm{nom}}$) can be used to represent, respectively, low, medium and high performance. In systems with less or more stringent requirements, it is straightforward to shift the detection-currency scale as appropriate.

Table 1: Nominal operating points

Quality	P_d	P_f	$\mathrm{J}_{\mathrm{dB}}^{\mathrm{asy}}$	$\mathrm{J}_{\mathrm{dB}}^{\mathrm{nom}}$
low	0.5	10^{-4}	5.7 dB	5 dB
medium	0.7	10^{-8}	7.9 dB	8 dB
high	0.9	10^{-16}	9.8 dB	10 dB

3. MODELING J-DIVERGENCE IN SONAR APPLICATIONS

Evaluating detection performance requires defining statistical models of the signal and noise as well as the detector itself. In this paper only the basic case of Gaussian bandpass noise and three standard signal models are considered with a focus on detectors formed from the instantaneous intensity, which include quadrature matched filters (QMFs) and energy detectors (EDs). It is also assumed that normalization by the background noise power is perfect.

The most common signal models employed in sonar analysis are the deterministic and Gaussian-fluctuating signals. Spanning the two is the Rician signal [5, Sect. 7.5.3], for which the instantaneous intensity is proportional to a non-central-chi-squared-distributed random variable with two degrees of freedom,

$$\frac{2}{1+\rho s}T \sim \chi^2_{2,\frac{2(1-\rho)s}{1+\rho s}}, \tag{5}$$

where s [unitless] is the SNR after the coherent portion of detection processing (i.e., S^{d} in [5, Sect. 2.3]). The parameter ρ is the ratio of the Gaussian-random signal power to the total signal power. Setting $\rho = 1$ produces a Gaussian-fluctuating signal so (5) simplifies to T being exponential with mean $1 + s$, whereas a deterministic signal is obtained when $\rho = 0$.

Similar to the traditional metrics, the results for a Gaussian-fluctuating signal in Gaussian noise are accessible: the J-divergence for a single instantaneous intensity is simply

$$J = \frac{s^2}{1+s}. \tag{6}$$

This suggests that performance is proportional to SNR when it is large (e.g., for a matched filter) and proportional to the square of SNR when it is small (e.g., a single bin in an energy detector). Although forward models can be evaluated numerically for the other signal models, developing approximations to this one-to-one non-linear mapping from SNR to J is straightforward (e.g., see those presented in Table 2, which are accurate to ≈ 0.1 dB). For example, an approximation for the deterministic signal in Gaussian noise is

$$J \approx \frac{2s^2}{2+s}\left[1 - \frac{1}{5.8}\exp\left\{\frac{-|S_{\mathrm{dB}} - 6.5|^{1.83}}{140}\right\}\right], \tag{7}$$

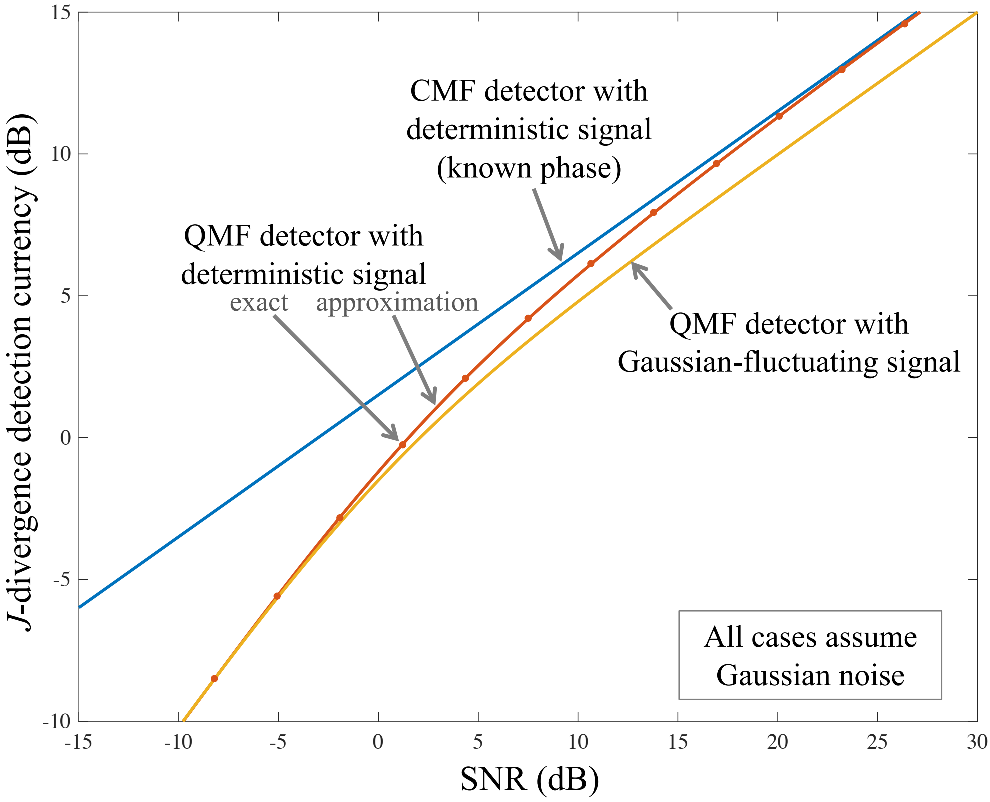

Figure 1: J-divergence detection currency as a function of SNR for the most common detection scenarios in Gaussian noise.

where $S_{\mathrm{dB}} = 10\log_{10} s$ is the SNR in decibels. The validity of the approximation is demonstrated in Fig. 1 where J-divergence is shown as a function of SNR for the deterministic and Gaussian-fluctuating signals. Because the term in brackets in (7) tends to one at very high and low SNR, the preceding $2s^2/(2+s)$ term drives the results at the extremes. This implies performance at low SNR will be $J \approx s^2$, which is the same as a Gaussian-fluctuating signal—the determinism of the amplitude does not counter the deleterious effects of the noise. At the other extreme, however, performance tends to $J \to 2s$, which is the blue line in Fig. 1 and represents the performance of a coherent matched filter (CMF). It is seen here that there is only a vanishingly small penalty as SNR increases when using a QMF because the bulk phase of a deterministic signal is unknown. This result also provides a compact explanation for why Albersheim's equation [6], which was derived for deterministic signals, can be used to obtain DT for detecting Gaussian-fluctuating signals with energy detectors (the SNR in each bin is small) but generally not for matched filters where the SNR in a single instantaneous intensity is higher.

3.1. ENERGY DETECTORS AND MULTIPLE MEASUREMENTS

The forward models in (6), (7), and Table 2 represent the J-divergence achieved from a single instantaneous intensity. When a detector sums multiple independent instantaneous intensities, as is the case for energy detectors and post-matched-filter integrators, the total J-divergence is the sum of that achieved by the individual measurements. The discussion here focuses on the energy-detector variants presented in [5, Sects. 9.2.5 & 9.2.6]. As noted in Sect. 2, summing J-divergences assumes the measurements are combined by an optimal detector. This approach is appropriate in the rare cases where the optimal detector can be implemented and is a good approximation when using an Eckart filter, which is only slightly sub-optimal.

In the more common case where the intensity samples are normalized by their noise power before summing them (i.e., a noise-normalized energy detector; NN-ED), a better approximation is

$$J = MJ_0, \tag{8}$$

where M is the number of independent intensities in the sum ($M \approx$ time-bandwidth product) and J_0 is the J-divergence at the average-linear-quantity SNR. When converting this to decibels, the performance is that achieved by a single instantaneous intensity plus $5\log_{10} M$,

$$\mathrm{J}_{\mathrm{dB}} = 5\log_{10} J_0 + 5\log_{10} M \quad \text{[units: dB]}. \tag{9}$$

This comports with our expectations for performance when detectors combine information incoherently. In contrast to the traditional techniques (where this is only seen asymptotically for large M in inverse models such as Albersheim's equation), this relationship exists explicitly in the forward modeling of J-divergence for all M and greatly simplifies inverse modeling.

Table 2: Equations for forward and inverse modeling of J-divergence; all cases assume a background of Gaussian noise; errors in the approximations are generally less than 0.1 dB

Definitions

$J =$ the linear-quantity J-divergence

$S_{\mathrm{dB}} = 10\log_{10} s$ is the SNR [dB] in a single instantaneous intensity

$\mathsf{DJ} =$ single-intensity SNR [dB] required to achieve a detection currency of $\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}} = 5\log_{10} J^{\mathrm{des}}$

$\Phi(z) =$ the standard-normal cumulative distribution function

Gaussian-fluctuating signal

$J = s^2/(1+s)$

$\mathsf{DJ} = 2\,\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}} - 3 + 10\log_{10}\left[1 + \sqrt{1 + 4\cdot 10^{-\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}}/5}}\right]$

Deterministic signal

$J \approx \frac{2s^2}{2+s}\left[1 - \frac{1}{5.8}\exp\left\{\frac{-|S_{\mathrm{dB}}-6.5|^{1.83}}{140}\right\}\right]$

$\mathsf{DJ} \approx 2\,\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}} - 3 + 10\log_{10}\left[1 + 2\cdot 10^{-\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}}/10}\right] + 10\log_{10}\left[1 - 0.08\exp\left\{\frac{-|\,\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}}-5.2|^{1.8}}{40}\right\}\right]$

Rician signal $\quad\left[\text{mixture of above results using } \mathcal{P} = \Phi([\log(\rho) - \mu_r]/\sigma_r)/\Phi(-\mu_r/\sigma_r)\right]$

$J \approx \mathcal{P} J_{\mathrm{GAU}} + (1-\mathcal{P})J_{\mathrm{DET}}$ with $\mu_r = -\log\left(4.4 + 10^{S_{\mathrm{dB}}/10}\right)$ & $\sigma_r = 0.6\Phi\left(\frac{S_{\mathrm{dB}}-10}{6}\right) + 1.1$

$\mathsf{DJ} \approx \mathcal{P}\,\mathsf{DJ}_{\mathrm{GAU}} + (1-\mathcal{P})\mathsf{DJ}_{\mathrm{DET}}$ with $\mu_r = -\log\left[5 + 0.5\cdot 10^{\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}}/5}\right]$ & $\sigma_r = 0.3\Phi\left(\frac{\mathrm{J}^{\mathrm{des}}_{\mathrm{dB}}-10}{5}\right) + 1.2$

3.2. INVERSE MODELING FOR THE DESIGN SNR

Particularly when combining multiple dissimilar measurements, models of PDM can be complicated to evaluate and although those for PFM are less challenging, they need to be solved for the decision threshold. This has driven the use of inverse modeling after simplification of the problem to fit tools such as Albersheim's equation [6], which provides the SNR required to achieve a desired $(\mathrm{PDM}, \mathrm{PFM})$ operating point when summing M independent envelopes containing a deterministic signal in Gaussian noise. Albersheim's equation is reasonably accurate when summing intensity samples [7], which makes it popular for assessing energy detectors.

For simple cases, such as the Gaussian-fluctuating signal in Gaussian noise, the detection-currency equivalent is obtained by solving the forward model in (6) for s and converting to decibels, which produces a design SNR of

$$\mathsf{DJ} = 2\,\mathrm{J}_{\mathrm{dB}}^{\mathrm{des}} - 3 + 10\log_{10}\left[1 + \sqrt{1 + 4\cdot 10^{-\mathrm{J}_{\mathrm{dB}}^{\mathrm{des}}/5}}\right] \qquad \text{[units: dB]}, \tag{10}$$

for a single instantaneous intensity, where $\mathrm{J}_{\mathrm{dB}}^{\mathrm{des}} = 5\log_{10} J^{\mathrm{des}}$ is a detection-currency specification. DJ is a direct replacement for the DT (detection threshold) term in the sonar equation that starts with a performance specification in J-divergence detection currency (e.g., via Table 1).

As was done in the forward modeling, simple approximations to DJ were obtained for the Rician and deterministic signals (see Table 2). These approximations require at most evaluation of the standard normal CDF, $\Phi(z)$. When summing M independent intensity samples, the average design SNR achieving the performance specification is simply obtained by evaluating DJ in (10) or Table 2 at $\mathrm{J}_{\mathrm{dB}}^{\mathrm{des}} - 5\log_{10} M$. These results make evaluation of the design SNR for detecting the three basic signal models in Gaussian noise with an intensity-integrating detector essentially as accessible as Albersheim's equation.

Table 3: Comparison of design SNRs: DT *and* DJ *[units: dB].*

Signal	Detector	M	Low-quality OP			Medium-quality OP			High-quality OP		
			DT	$\mathsf{DJ}_{\mathrm{asy}}$	$\mathsf{DJ}_{\mathrm{nom}}$	DT	$\mathsf{DJ}_{\mathrm{asy}}$	$\mathsf{DJ}_{\mathrm{nom}}$	DT	$\mathsf{DJ}_{\mathrm{asy}}$	$\mathsf{DJ}_{\mathrm{nom}}$
DET	CMF	1	8.4	8.4	7.0	12.7	12.7	13.0	16.5	16.5	17.0
DET	QMF	1	9.4	9.9	8.8	13.2	13.7	13.9	16.9	17.1	17.5
GAU	QMF	1	10.9	11.7	10.4	17.0	15.9	16.1	25.4	19.6	20.0
GAU	NN-ED	2	7.8	8.9	7.7	12.7	13.0	13.2	18.8	16.6	17.1
GAU	NN-ED	5	4.5	5.5	4.4	8.4	9.3	9.5	12.8	12.8	13.2
GAU	NN-ED	10	2.3	3.1	2.1	5.8	6.6	6.8	9.5	10.0	10.4
GAU	NN-ED	50	-2.0	-1.7	-2.5	0.7	1.2	1.4	3.4	4.0	4.4
GAU	NN-ED	100	-3.8	-3.5	-4.3	-1.2	-0.8	-0.7	1.3	1.8	2.1
GAU	NN-ED	500	-7.6	-7.4	-8.2	-5.2	-5.0	-4.9	-3.0	-2.8	-2.5
GAU	NN-ED	10^3	-9.1	-9.0	-9.8	-6.8	-6.7	-6.6	-4.7	-4.6	-4.3

4. EXAMPLE COMPARISON OF DESIGN SNR

To illustrate the similarities and differences between these approaches, the design SNRs required to achieve the low, medium and high levels of performance described in Table 1 are compared in Table 3. The values of DT were obtained by numerically inverting the PFM and PDM equations. In contrast, DJ was evaluated using the equations in Table 2. The most direct comparison is between DT and $\mathsf{DJ}_{\mathrm{asy}}$, where the value of $\mathrm{J}_{\mathrm{dB}}^{\mathrm{des}}$ was obtained from $(\mathrm{PDM}, \mathrm{PFM})$ using the asymptotic ROC in (3). As expected, the larger disparity at small M vanishes as $M \to \infty$. However, they are quite similar for the deterministic signal (DET) in the QMF and for the Gaussian-fluctuating signal (GAU) when $M \geq 5$ in the NN-ED. Given the inherent inaccuracies in sonar modeling, an exact equivalence is not necessary, suggesting detection currency is useful even when $M < 5$. The design SNRs labeled DJ_{nom} (blue text) were obtained using the nominal detection-currency specifications (i.e., 5, 8, & 10 dB). Their similarity to DT suggests that these qualitative performance levels will be useful in scenarios where a specific operating point is not paramount or not easily modeled (e.g., owing to inherent inaccuracies).

5. CONCLUSIONS

An alternative method for evaluating sonar detection performance, based on the J-divergence and described as a *detection currency*, has been proposed. Although its origins are more esoteric than the standard metrics, its application is easier than traditional methods, particularly when systems exploit multiple dissimilar measurements. As an information theoretic measure, it can be applied at different points in the signal and information processing chain, potentially improving prediction of system-level performance. As a single scalar performance metric on a decibel scale, this detection currency can be useful in optimizing signal processing algorithms or sonar systems and simplifies explanation of detection performance to non-specialists.

6. ACKNOWLEDGEMENTS

This work was sponsored by the Office of Naval Research; Code 321 Undersea Signal Processing.

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