Theoretical Model for Correlation Function of Wideband Signal Received by Two Vertical Arrays and Retrieval of Modal Dispersion Curves in Shallow Water

Marina Yarina\textsuperscript{a}, Boris Katsnelson\textsuperscript{a}, Oleg A. Godin\textsuperscript{b}

\textsuperscript{a}Department of Marine Geosciences, University of Haifa, Haifa, Israel
\textsuperscript{b}Department of Physics, Naval Postgraduate School, Monterey, CA, USA

M. Yarina, 199 Abba Khouchy ave., University of Haifa, Haifa, 3498838, Israel, Email: myarina@campus.haifa.ac.il

Abstract: The paper presents the results of theoretical modeling and numerical calculations of the cross-correlation function from a broadband source, both deterministic and random (for example, ship noise) in a shallow water waveguide. The signal is received by two synchronized vertical linear arrays (VLAs) located at some distance \(d\) from each other. The parameters of the problem correspond to the conditions of experiments carried out earlier by the authors in Lake Kinneret (Israel): depth 40 m, distance between the moving source and receiving antennas \(\sim 500–1500\) m, source spectrum in the low-frequency region \((30–300 \text{ Hz})\). An important feature of this area is the presence of a gas-saturated layer of sediments with a thickness \(\sim 1\) m. The CCF of the problem in this case is constructed in coordinates (wave number, frequency) and allows us to estimate dispersion curves for waveguide modes, including modes, trapped in the narrow gassy layer. Taking into account that receiving system is set of hydrophones and the corresponding expressions for CCF should be written in the matrix form. Analysis of CCF allows us to estimate parameters of sediments, more specifically thickness of gassy layer and gas concentration.

Keywords: noise correlation function, geoacoustic inversion, shallow water
1. INTRODUCTION. STATEMENT OF THE PROBLEM

Existence of gas saturated layer in sediments is rather widespread situation both in salty and fresh-water aquatic systems, it is the subject of acoustical research [1] for possible estimation of their parameters in seas and lakes [2]. We can examine the following scenario of sound propagation in shallow water (Figure 1) which was used in Lake Kinneret (Israel) [3]. Two vertical synchronized arrays (VLAs) are fixed at some distance $\Delta r$ between them. Wideband source is moving along the line joining arrays (axis $r$) for a certain duration. The source could be either shipping noise or a controlled signal.

![Fig.1. Layout of the experiment. East and West VLA are denoted.](image)

We assume that the water layer has the depth of $H$ and is characterized by some sound speed profile $c_1(z)$. Below the water layer, there is a thin ($h \sim 1$ m) layer of gas-saturated sediment with a low sound speed $c_2 << c_1$ that depends on the gas concentration. A homogeneous half space (bottom) is placed below gassy layer and has a sound speed $c_b$.

2. CORRELATION FUNCTION

We begin by considering the sound signal originating from the source, which possesses a spectrum $S(\omega)$, located at the point measured at the distance $r$. We perform modal decomposition on both arrays, denoted as West and East, placed at the distances $r$ and $r + \Delta r$ respectively.

$$P_E(\omega, r, z) = S \sum_n \frac{\psi_n(z)}{\sqrt{q_n r}} \psi_n(z) \exp[i q_n r]; \quad \text{(1)}$$

$$P_W(\omega, r + \Delta r, z) = S \sum_n \frac{\psi_n(z)}{\sqrt{q_n (r + \Delta r)}} \exp[i q_n (r + \Delta r)]. \quad \text{(2)}$$

The waveguide modes (eigenfunctions) and eigenvalues denoted as $\psi_n(z)$, $q_n$ are depending on frequency. These modes are solutions of Eq. (3) subject to their respective boundary conditions.

$$\left[ \frac{d^2}{dz^2} + (k^2 - q_n^2) \right] \psi_n(z; \omega) = 0 \quad \text{(3)}$$

$$\psi_n(z; \omega)|_{z=0} = 0, \quad [\psi_n + g \frac{d\psi_n}{dz}]|_{z=H} = 0, \quad [\psi_n + g_b \frac{d\psi_n}{dz}]|_{z=H+h} = 0.$$
where \( g \) and \( gb \) parameters, characterizing properties of boundaries at \( z=H \) and \( z=H+h \), where sound speed profile is shown on Fig.2.

![Fig.2 Sound speed profile in a waveguide.](Image)

Our goal is to find parameters \( (q_n(\omega)) \) of the modes and subsequently estimate bottom properties, particularly the thickness of the layer and its sound speed using experimental data. Let’s introduce function and values \( \psi_q(z; \omega) \), (where \( q \) is continuous variable) satisfying differential equation with one boundary condition on the surface

\[
\left[ \frac{d^2}{dz^2} + (k^2 - q^2) \right] \psi_q(z; \omega) = 0, \quad \psi_q(z; \omega) \bigg|_{z=0} = 0. \tag{4}
\]

Remark, that in real situation the second (and the third) boundary conditions (parameters of sediment) are unknown for us, we will assume at this case \( q < \omega / c^\text{layer}_{\text{min}} \), where \( c^\text{layer}_{\text{min}} \sim 100 \text{ m/s} \), lower limit is determined by the sound speed in the half-space.

The calculations for theoretical sound field \( P^E(r, z; \omega) \) and \( P^W(r + \Delta r, z; \omega) \) (Eq. 1, 2) are treated as synthetic experimental data in our study.

Equations 5 and 6 allow us to calculate the modal amplitudes for each antenna, depending on the frequency, for any given value of \( q \) and mode number.

\[
A_E(q, \omega) = \int_0^H P^E(r, z; \omega) \psi_q(z; \omega) \, dz = S \sum_n \frac{\psi_n(x_z)}{\sqrt{q_n}} \int_0^H \psi_n(z) \psi_q(z; \omega) \, dz = S \sum_n \frac{\psi_n(x_z)}{\sqrt{q_n}} \exp[iq_n r] B_{nq}(\omega) \tag{5}
\]

\[
A_W(q, \omega) = S \sum_n \frac{\psi_m(x_z)}{\sqrt{q_m(r+\Delta r)}} \exp[iq_m(r+\Delta r)] B_{mq}(\omega) \tag{6}
\]

Equations 4 and 5 incorporate the matrix element \( B_{nq} \) (Eq.6), which takes on the value of a delta function operator if \( q \) corresponds to one of the solutions of Equation 3. Otherwise, it assumes different numerical values.

\[
B_{nq}(\omega) = \int_0^H \psi_n(z) \psi_q(z; \omega) \, dz. \tag{7}
\]

We initiate the construction of the correlation function by convolving matrices of modal amplitudes that are dependent on both frequency and horizontal wavenumber.

\[
A_E(q, \omega)A_W(q, \omega)^* = S^2 \sum_{n,m} \frac{\psi_n(x_z)\psi_m(x_z)}{\sqrt{q_m(r+\Delta r)}} \exp[-iq_m(r+\Delta r) + iq_n r] B_{nq}(\omega) B_{mq}(\omega)^* \tag{8}
\]

Subsequently, we perform an average over the time of source movement for Equation 8.
< A_E(q, \omega) A_W(q, \omega)^* > = S^2 \sum_n \frac{\psi_n^2(z_s)}{q_n r} \exp[-i q_n \Delta r] B_{nq}(\omega)^2
\tag{9}

Correlation function constructed using compensation of the phase shift between two arrays has the following form:
\begin{equation}
K_{12}(q, \omega) = S^2 \sum_n \frac{\psi_n^2(z_s)}{q_n r} \exp[i(q_n - q) \Delta r] B_{nq}(\omega)^2
\end{equation}
\tag{10}

On the plane \((q, \omega)\) correlation function is a sum of oscillating functions, if \(q = q_m\), then \(B_{nm}(\omega) \sim 0, B_{mn}(\omega) \sim 1\). However, the correlation function does not exhibit exact periodicity due to its integration over a specific depth range.

3. DISCUSSIONS

Fig.3. Left: Cross-correlation function dependency on frequency and horizontal wavenumber \(q\). Black lines denote wavenumbers \(k\) for low sound speed layer, water and bottom half space. White dots are the real roots of Eq.3. Right: matrix element Eq.6 calculated for mode number 4. Both colour scales are logarithmic. TM means Trapped modes, WM – water modes and LM – leaking modes. That denotes parts of \((q, \omega)\) plane.

In the Fig.3 beamforming correlation function \(K_{12}(\omega, q)\) is presented, calculated using parameters, mentioned above. White lines denote theoretical dispersion curves, which constructed at the maximums of \(K_{12}(\omega, q)\). They are shown in areas: (1) – WM - between straight lines \(q = c_1 \omega\) and \(q = c_2 \omega\), these curves correspond to modes, propagating in water layer, number of these modes depend on frequency and with increasing of frequency given mode transforms in mode trapped by gassy layer, they is shown in area (2) – TM-, which is between \(q = c_1 \omega\) and \(q = c_2 \omega\). In the Fig. 4 transformation of mode, propagating in water layer (WM) into mode, trapped in gassy layer (TM). It is mode 3, having “critical” frequency \(\sim 100\) Hz. On Fig.3 (left) 5 modes consequently transform into trapped modes, each one at the corresponding “critical” frequency. Lower area (LM) corresponds to waves leaking from waveguide, it is denoted as LM (leaking modes).
Fig.4. Behavior of eigenfunction of mode number 3 changing with frequency while transitioning into propagation through sedimentary layer. Blue line shows part of eigenfunction in water layer, red in sedimentary layer.

Remark, that in real situation we are dealing with discrete set of hydrophones on VLA, so we should use summation instead of integration for construction of the correlation function. In this case expression for CCF will be written in the matrix form. The corresponding amplitudes are as follows (Eq.11, 12):

\[ A^E(\omega, r, q) = \sum_{j=1}^{N} P^E(\omega, r, z_j) \hat{\psi}(\omega, z_j, q) \Delta z; \tag{11} \]

\[ A^W(\omega, r + \Delta r, q) = \sum_{j=1}^{N} P^W(\omega, r + \Delta r, z_j) \hat{\psi}(\omega, z_j, q) \Delta z. \tag{12} \]

Therefore, after discretizing the modal amplitudes, the correlation function will have the following form:

\[ K_{12}(\omega, q) = \text{Re}(A^E(\omega, q, r)A^E(\omega, q, r + \Delta r)^* \exp(iq\Delta r)). \tag{13} \]

Presenting the algorithm for constructing the correlation function using a constant kernel matrix \( P \) and a set of vectors \( s \) as changing parameters is simpler. The exponential term acts as a constant multiplier for phase compensation.

\[ K_{12}(\omega, q) = \text{Re}(s \cdot P \cdot s^T \exp (iq\Delta r)); \tag{14} \]

\[ s = \{ \psi(\omega, z_1, q), \psi(\omega, z_2, q) ... \psi(\omega, z_N, q) \} \tag{15} \]

Elements of matrix \( P \) (eq.13) are calculated as follows:

\[ P_{ij} = \langle P^E(\omega, z_i, r)P^W(\omega, z_j, r + \Delta r)^* \rangle. \tag{16} \]

In the Fig.5 we can see CCF, constructed using expression (13) where it is supposed that VLAs have 10 hydrophones each, spaced by 3 m. We can see that structure of the CCF is a little less expressed in comparison with the Fig.3 (left), however dispersion curves can be retrieved both in areas of WM and TM, that in principle can allow us to estimate thickness of gassy layer and sound speed \( c_2 \) connecting with gas concentration.
Fig. 5. Cross-correlation function dependency on frequency and horizontal wavenumber $q$, calculated for discrete antenna. Black lines denote wavenumbers $k$ for low sound speed layer, water and bottom half space. White dots are the real roots of Eq.3.

4. CONCLUSION

In conclusion remark that methodology, using the corresponding processing of wideband signal, including shipping noise can give us information about sediment properties. Simultaneously, there are challenges associated with the retrieval of dispersion curves in the context of gas-saturated layers. One major obstacle is the presence of significant attenuation within the gassy layer, which can affect the accuracy of measurements. Additionally, the amplitudes of the "trapped" modes within this layer can be exceptionally small. Nevertheless, the "critical" frequency at which these modes cease to exist provides valuable information regarding the thickness and sound speed of the layer.

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