J-divergence detection currency after thresholding a Rician signal in Gaussian noise

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Abstract: *Sonar performance modeling traditionally entails the use of single-measurement probabilities of detection and false alarm as metrics. Extension to systems utilizing multiple measurements is often complicated enough that the scenarios are simplified to fit the single-measurement tools, thereby approximating system-level performance. By allowing an approximate representation of single-measurement detection performance in the form of J-divergence detection currency (JDC), a more accurate assessment of system-level performance can be obtained across multiple measurements. As a measure of detection performance potential, JDC can be evaluated at different points in the signal and information processing chain. After coherent detection processing, a common step in multiple-measurement systems is the thresholding of individual measurements before they are combined in a final detection decision. For example, this typically occurs when combining measurements across waveforms in active sonar or across sensors in distributed systems. The focus of this paper is on techniques for evaluating JDC after thresholding for an integrated-intensity detector operating on the standard sonar signal models (Gaussian-fluctuating, deterministic, and the more general Rician model) in Gaussian noise.*

Keywords: *detection performance, J-divergence, Rician signal*

1. INTRODUCTION AND BACKGROUND

Traditional techniques for modeling sonar performance typically exploit the sonar equation to obtain the signal-to-noise power ratio (SNR) and then relate SNR to single-measurement probabilities of detection and false alarm (i.e., P_d and P_f). Applications employing multiple measurements are often simplified to fit within the context of the single-measurement tools, potentially leading to inaccurate predictions of system performance. An alternative approach using the J-divergence as a general representation of *detection currency* was presented in [1]. At the expense of an approximate relationship to the detection probabilities, multiple independent measurements are easily incorporated by summing their J-divergences. Owing to a data processing inequality (processing of data can only maintain or decrease J-divergence, but not increase it), the technique allows evaluation at different points in the signal and information processing chain, with an interpretation as representing detection performance potential. The focus of this paper is on carrying the analysis of J-divergence through the thresholding process commonly found in multiple-measurement systems. For example, echoes from multiple waveforms in active sensing or measurements from different sensors in a distributed system typically require the individual measurements to exceed a local threshold before they are combined to form a final detection decision.

The J-divergence (J) is a one-to-one non-linear mapping from the SNR obtained through the sonar equation to a scalar measure of detection performance. For example, from [1], an instantaneous intensity sample containing a Gaussian-fluctuating signal in Gaussian noise results in $J = s^2/(1+s)$ where s [unitless] is the linear-quantity SNR. This implies detection performance is quadratically related to SNR when $s \ll 1$ and tends to be linear when $s \gg 1$. Accounting for thresholding has little effect when the SNR is high enough for P_d to be close to one, but will significantly reduce J when the individual measurement is unlikely to be detected. Carrying this effect into the analysis of multiple-measurement detection systems produces a more accurate prediction of system-level performance, particularly at low SNR.

The primary contributions of this paper are techniques for the evaluation of J-divergence after thresholding for an integrated-intensity detector (e.g., after quadrature matched filtering or in an energy detector) observing one of the standard sonar signal models (Gaussian-fluctuating, deterministic, or Rician) in Gaussian noise. The J-divergence after thresholding is introduced and examined using the Gaussian-fluctuating signal in Sect. 2 and related to the Kullback-Leibler divergences of the excess over the threshold. Extending the results to the Rician signal model (which contains the deterministic and Gaussian-fluctuating signals) requires an approximation. In Sect. 3, the integrated-intensity decision statistic is modeled by a generalized gamma (GG) distribution with the parameters obtained by matching moments with the Rician model. To avoid simultaneous solution of the moment equations, an approximation to the GG power parameter is presented, which simplifies the problem to the solution of a single non-linear moment equation. The J-divergence after thresholding is then derived for GG distributions in Sect. 4 and used in Sect. 5 to demonstrate that the approximation error for the Rician model is adequately low over a wide range of parameters.

2. J-DIVERGENCE AFTER THRESHOLDING

The J-divergence [2] is a distributional distance measure useful in representing detection performance when applied to the probability density functions (PDFs) of a decision statistic under the noise-only (H_0) and signal-present (H_1) hypotheses. If T is the detector decision statistic and $f_0(t)$ and $f_1(t)$ are the PDFs under the two hypotheses, the J-divergence is

$$J = \int_{-\infty}^{\infty} [f_1(t) - f_0(t)] \log\left[\frac{f_1(t)}{f_0(t)}\right] \, \mathrm{d}t. \tag{1}$$

As described in [1], this can be converted to a logarithmic quantity, termed the J-divergence detection currency (JDC), through $\mathrm{JDC} = 5\log_{10}(J)$ [units: dB], as if it were a ratio of squared intensities. The J-divergence is combined over multiple independent measurements by summing the individual linear quantities, with the result representative of an optimal processor (e.g., using a likelihood-ratio detector).

In many multiple-measurement applications, individual measurements are thresholded prior to being combined. By treating the thresholding process as producing either a null state (threshold not exceeded) or providing a measurement representative of the threshold exceedance, the J-divergence after thresholding can be shown to be

$$J = [P_d - P_f] \log\left[\frac{1 - P_f}{1 - P_d}\right] + \int_{h}^{\infty} [f_1(t) - f_0(t)] \log\left[\frac{f_1(t)}{f_0(t)}\right] \, \mathrm{d}t, \tag{2}$$

where $P_d = \Pr\{T \geq h | H_1\}$, $P_f = \Pr\{T \geq h | H_0\}$, and h is the local threshold. This form is useful for analytical evaluation when the data PDFs and the log likelihood ratio combine in the integrand to result in simple forms. It is also a convenient form for numerical evaluation.

As an example, consider a decision statistic formed by integrating (summing) M independent and identically distributed intensities, as might occur in an energy detector in passive sensing or after post-matched-filter integration in active sensing. If each intensity contains a Gaussian-fluctuating signal in Gaussian noise with a linear-quantity SNR of s [unitless], the decision statistic is gamma distributed with shape parameter M and scale parameter $1 + s$. Thresholding the decision statistic at h yields J-divergence

$$\begin{aligned} J = [P_d - P_f] \log\left[\frac{1 - P_f}{(1 - P_d)(1 + s)^M}\right] + Ms\left[P_d - \frac{P_f}{1 + s}\right] \\ + \frac{h^M e^{-h}}{\Gamma(M)} \frac{s}{(1 + s)} \left[\frac{e^{hs/(1+s)}}{(1 + s)^{M-1}} - 1\right], \end{aligned} \tag{3}$$

where $P_f = 1 - \tilde{\gamma}(h; M)$, $P_d = 1 - \tilde{\gamma}(h/[1 + s]; M)$, and $\tilde{\gamma}(t; \alpha)$ is the incomplete gamma function normalized by $\Gamma(\alpha)$ [4, pg. 293]. Removing the thresholding by letting $h \to 0$ so P_d and $P_f \to 1$ yields $J = Ms^2/(1 + s)$ from the second term in (3), which produces the result found in [1, eq. 6] when $M = 1$ (representing a single intensity sample).

The simplicity of this result makes it quite useful. However, signals exhibiting greater consistency from observation to observation are better represented by a deterministic or Rician signal. Similar to evaluating J-divergence before thresholding, the Bessel functions found in the intensity PDFs of these signals in Gaussian noise complicate a direct solution. However, as presented in Sect. 3, an accurate approximation that is straightforward to evaluate can be obtained using a generalized gamma distribution.

2.1. RELATIONSHIP TO KULLBACK-LEIBLER DIVERGENCES

For some PDFs, the distribution of the excess over the threshold, $Z = T - h$ given T exceeds h can be described simply. For example, when T is exponentially distributed with mean λ, $Z = T - h$ is also exponentially distributed with mean λ when conditioning on $T \geq h$. This can facilitate evaluation of the J-divergence after thresholding in (2) by describing the integral using the Kullback-Leibler divergences (KLDs) of Z in conjunction with P_f and P_d.

First note that the PDF of Z (given $T \geq h$) under H_0 and H_1 is easily described by the PDF of the unthresholded data and the appropriate exceedance probability,

$$\tilde{f}_0(z) = f_0(z+h)/P_f \text{ and } \tilde{f}_1(z) = f_1(z+h)/P_d \text{ for } z \geq 0. \tag{4}$$

This allows rephrasing the integral in (2) as $[P_d - P_f]\log[P_d/P_f] + P_d\tilde{I}_{1:0} + P_f\tilde{I}_{0:1}$, where $\tilde{I}_{i:j} = \int_{-\infty}^{\infty} \tilde{f}_i(z)\log\left[\tilde{f}_i(z)/\tilde{f}_j(z)\right]\mathrm{d}z$ is the KLD [3] formed using the distributions of the threshold exceedance. The J-divergence after thresholding,

$$J = [P_d - P_f]\log\left[\frac{P_d[1-P_f]}{P_f[1-P_d]}\right] + P_d\tilde{I}_{1:0} + P_f\tilde{I}_{0:1}, \tag{5}$$

can then be characterized as comprising a binary component and contributions accounting for the excess over the threshold after moderation by the probability the event occurs.

3. APPROXIMATING A RICIAN INTEGRATED INTENSITY WITH A GENERALIZED GAMMA DISTRIBUTION

The most common signal models employed in sonar analysis are the deterministic and Gaussian-fluctuating signals. Spanning the two is the Rician signal [4, Sect. 7.5.3], for which the integrated instantaneous intensity (T) is proportional to a non-central-chi-squared-distributed random variable with $2M$ degrees of freedom,

$$\frac{2}{1+\rho s}T \sim \chi^2_{2M,\delta} \tag{6}$$

where M and s are as described in Sect. 2, $\rho \in [0,1]$ is the ratio of the Gaussian-random signal power to the total signal power, and $\delta = 2M(1-\rho)s/(1+\rho s)$ is the non-centrality parameter. Setting $\rho = 0$ produces a deterministic signal and a Gaussian-fluctuating signal is obtained when $\rho = 1$ (with simplification to the gamma distribution described in Sect. 2).

The PDF of the non-central-chi-squared distribution contains a Bessel function that complicates a direct analysis of the J-divergence. For the purposes of evaluating J-divergence after thresholding, the Rician signal model can be approximated with a generalized gamma (GG) distribution. If X is gamma distributed with shape α and unit scale, then $T = \beta X^{1/c}$ follows the GG distribution and has PDF

$$f_T(t) = \frac{c t^{c\alpha-1} e^{-(t/\beta)^c}}{\Gamma(\alpha)\beta^{c\alpha}} \text{ for } t \geq 0, \ \alpha > 0, \ \beta > 0, \text{ and } c > 0. \tag{7}$$

When the GG power parameter $c = 1$, T is gamma distributed with shape parameter α and scale parameter β. The Weibull distribution can also be seen to arise when $\alpha = 1$.

The difficulty in exploiting the GG model lies in obtaining the parameters (α, β, c) when a signal is present. Choosing the parameters that match the first three moments of T (i.e., $\mu'_k = E[T^k]$ for $k = 1$, 2, & 3) between the Rician and GG models results in the moment equations

$$\mu'_1 = M(1+s) = \frac{\beta\,\Gamma(\alpha+1/c)}{\Gamma(\alpha)}, \tag{8}$$

$$\mu'_2 = M^2(1+s)^2 + M(1+\rho s)(1-\rho s+2s) = \frac{\beta^2\,\Gamma(\alpha+2/c)}{\Gamma(\alpha)}, \text{ and} \tag{9}$$

$$\mu'_3 = M(M+1)(M+2)(1+\rho s)^3\left[1+3d+\frac{3Md^2}{M+1}+\frac{M^2d^3}{(M+1)(M+2)}\right] = \frac{\beta^3\,\Gamma(\alpha+3/c)}{\Gamma(\alpha)},$$

where $d = (1-\rho)s/(1+\rho s)$. These equations need to be solved for α, β, and c as functions of ρ, s, and M. For a Gaussian-fluctuating signal in Gaussian noise (i.e., when $\rho = 1$), the decision statistic is gamma distributed so $c = 1$, $\alpha = M$, and $\beta = 1+s$. Setting $s = 0$ then produces the parameters under H_0 (i.e., β reduces to one). However, with both α and c trapped inside gamma functions, the solution for signals with $\rho < 1$ must be obtained iteratively. As this is prohibitive in many applications, an approximation only requiring the solution of a single non-linear equation is presented in Sect. 3.1.

Once the parameters of the GG model are obtained under the two hypotheses, the J-divergence after thresholding can be evaluated as described in Sect. 4.

3.1. APPROXIMATE MOMENT MATCHING

After evaluating c though precise moment matching for a wide range of (ρ, s, M), it was found that using the following approximation was adequate for evaluating J-divergence after thresholding,

$$c \approx 1 + \exp\left\{-0.5 - \exp\left\{-1.3 - \frac{1}{2M} + \rho\left(2.8 + \frac{0.6}{M}\right)\right\}\right\}\Phi(S_{\mathrm{dB}}/5)\,, \tag{10}$$

where $\Phi(z)$ is the standard normal cumulative distribution function. The SNR used in (10) is the logarithmic quantity, $S_{\mathrm{db}} = 10\log_{10}(s)$ [units: dB].

Given c, the moment ratio $\mu'_2/[\mu'_1]^2$ is straightforward to solve numerically for α. When a root-finding routine is not available, a Newton-Raphson iteration can be implemented as follows. Let $A = \log(\alpha)$ so $\alpha = e^A$ and initialize A to $\log M$. The solution (in terms of A) can then be obtained by the iteration

$$A \leftarrow A - \frac{\log\left(\frac{\mu'_2}{[\mu'_1]^2}\right) + 2\log\Gamma\left(e^A+\frac{1}{c}\right) - \log\Gamma\left(e^A+\frac{2}{c}\right) - \log\Gamma\left(e^A\right)}{e^A\left[2\psi\left(e^A+\frac{1}{c}\right) - \psi\left(e^A+\frac{2}{c}\right) - \psi(e^A)\right]}, \tag{11}$$

where $\psi(\cdot)$ is the digamma function and $\mu'_2/[\mu'_1]^2 = 1+(1+\rho s)(1-\rho s+2s)/[M(1+s)^2]$ for the Rician model. The number of iterations required to achieve a specified error tolerance was seen to be decreasing in M and ρ and increasing in SNR (i.e., the combination of $M = 1$, $\rho = 0$ and high SNR requires the most iterations). Except at the highest SNRs (e.g., total SNR in excess of 30 dB), fewer than ten iterations were necessary to drive the relative error below 10^{-6}. Given α and c, the scale parameter β is obtained from (8) through $\beta = M(1+s)\Gamma(\alpha)/\Gamma(\alpha+1/c)$.

4. J-DIVERGENCE OF THRESHOLDED GENERALIZED GAMMA DISTRIBUTIONS

The J-divergence after thresholding can be obtained using the general form described in (2). Using the GG PDFs under H_0 and H_1 results in terms that can be described as partial moments and partial log moments. The upper partial pth-power moment for the GG distribution is

$$\int_h^\infty t^p f_T(t)\,\mathrm{d}t = \beta^p \int_{h'}^\infty \frac{z^{\frac{p}{c}+\alpha-1}e^{-z}}{\Gamma(\alpha)}\,\mathrm{d}z = \beta^p \Gamma(\alpha+p/c, h')/\Gamma(\alpha) \tag{12}$$

where $h' = (h/\beta)^c$ and $\Gamma(\alpha, x) = \int_x^\infty z^{\alpha-1}e^{-z}\,\mathrm{d}z$ is the complementary incomplete gamma function. The upper partial log moment of the GG distribution can be simplified to

$$\int_h^\infty \log(t) f_T(t)\,\mathrm{d}t = \log(\beta)[1-F_T(h)] + \frac{g(h';\alpha)}{c}, \tag{13}$$

where

$$g(h;\alpha) = \int_h^\infty \frac{\log(z)z^{\alpha-1}e^{-z}}{\Gamma(\alpha)}\,\mathrm{d}z \tag{14}$$

is the upper partial log moment of a gamma distribution with shape parameter α and unit scale. As shown in Sect. 4.1, this can be evaluated by truncating an infinite summation.

Before proceeding to the J-divergence after thresholding, define the normalized complementary incomplete gamma function as $\tilde{\Gamma}(z;\alpha) = \Gamma(\alpha, z)/\Gamma(\alpha) = 1 - \tilde{\gamma}(z;\alpha)$, noting that it is simply one minus the normalized incomplete gamma function. Using the transformed threshold $h_i' = [h/\beta_i]^{c_i}$, for $i = 0$ and 1, then allows describing the single-measurement detection probabilities as

$$P_d = \tilde{\Gamma}(h_1';\alpha_1) \quad \text{and} \quad P_f = \tilde{\Gamma}(h_0';\alpha_0)\,, \tag{15}$$

where the subscript on each parameter defines the hypothesis under consideration. The J-divergence after thresholding for the GG model can then be described as

$$\begin{aligned} J = &(P_d - P_f)\log\left[\frac{c_1(1-P_f)\Gamma(\alpha_0)}{c_0(1-P_d)\Gamma(\alpha_1)}\right] + (c_0\alpha_0 P_d - c_1\alpha_1 P_f)\log(\beta_0/\beta_1) \\ &- \alpha_1\tilde{\Gamma}(h_1';\alpha_1+1) - \alpha_0\tilde{\Gamma}(h_0';\alpha_0+1) + (c_1\alpha_1 - c_0\alpha_0)\left[\frac{g(h_1';\alpha_1)}{c_1} - \frac{g(h_0';\alpha_0)}{c_0}\right] \\ &+ \tilde{\Gamma}(h_1';\alpha_1+c_0/c_1)\frac{\Gamma(\alpha_1+c_0/c_1)}{\Gamma(\alpha_1)}\left[\frac{\beta_1}{\beta_0}\right]^{c_0} + \tilde{\Gamma}(h_0';\alpha_0+c_1/c_0)\frac{\Gamma(\alpha_0+c_1/c_0)}{\Gamma(\alpha_0)}\left[\frac{\beta_0}{\beta_1}\right]^{c_1}. \end{aligned} \tag{16}$$

Despite the length of this result, each term is straightforward to evaluate.

4.1. NUMERICAL EVALUATION OF THE GAMMA PARTIAL LOG MOMENT

The upper partial log moment of the gamma distribution from (14), $g(h;\alpha)$, is required to evaluate the J-divergence after thresholding of the gamma, Weibull, and generalized gamma distributions. A numerical solution in the form of a truncated infinite summation can be obtained by noting that $\log(z)z^\alpha$ is equal to the derivative of $z^\alpha = e^{\alpha\log z}$ with respect to α. This allows describing $g(h;\alpha)$ according to

$$g(h;\alpha) = \frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial\alpha}\int_h^\infty z^{\alpha-1}e^{-z}\,\mathrm{d}z\right] = \frac{1}{\Gamma(\alpha)}\frac{\partial\Gamma(\alpha,h)}{\partial\alpha}, \tag{17}$$

which is one over $\Gamma(\alpha)$ times the derivative of the complementary incomplete gamma function with respect to its order. Taking the partial derivative of the series expansion of $\Gamma(\alpha, h)$ found in [5, pg. 178, eq. 8.7.3] with respect to α and using it in (17) results in

$$g(h;\alpha) = \psi(\alpha)[1-\tilde{\gamma}(h;\alpha)] - h^{\alpha}e^{-h}\sum_{k=0}^{\infty}\frac{h^k[\log(h)-\psi(\alpha+k+1)]}{\Gamma(\alpha+k+1)}, \tag{18}$$

where $\psi(\cdot)$ is the digamma function. When $h \to 0$, it is clear that the partial log-moment tends to the full log moment, which is simply $\psi(\alpha)$. Truncating the infinite summation at $K_{\max} = \lceil 5 + 9h^{0.6} \rceil$ was found to be sufficient for the cases considered here.

5. ERRORS IN APPROXIMATING J-DIVERGENCE FOR A RICIAN SIGNAL

The parameters defining the J-divergence after thresholding for a Rician signal model include: the SNR (s), the fraction of random signal power (ρ), the number of independent intensities being integrated (M) and the decision threshold (h). As described in Sect. 3.1, these are used to obtain GG parameters (α_1, β_1, c_1) when signal is present, with the noise-only parameters set to $\alpha_0 = M$, $\beta_0 = 1$, and $c_0 = 1$. The J-divergence after thresholding is then evaluated as described in Sect. 4. A precise value was obtained for comparison through numerical integration of (2).

The error in the logarithmic quantity (i.e., JDC $= 10\log_{10}(J)$ [units: dB]) was evaluated for values of SNR and decision threshold producing $P_f \in [10^{-16}, 0.1)$ and $P_d \in [0.1, 0.9999]$, with $\rho \in [0, 1]$ and $M \in [1, 100]$. The maximum absolute error is shown in Fig. 1 for $M = 1$ (solid lines) as a function of ρ and as a function of $1/M$ (dotted lines), with the maximum taken over the other parameters. Using approximate moment-matching from Sect. 3.1 in the GG model (reddish-brown lines in Fig. 1) provides essentially the same worst-case accuracy (< 0.1 dB) as using exact moments (blue lines), with significantly less computational effort. Although the Weibull model (gold lines) is better for a single intensity ($M = 1$) and nearly deterministic signal, it fails when integrating intensities (i.e., when $M > 1$).

6. CONCLUSIONS

Techniques for evaluating the J-divergence detection currency (JDC) after thresholding for the basic sonar signal models in Gaussian noise were presented for an integrated-intensity detector (e.g., after matched filtering or in an energy detector). Although the Gaussian-fluctuating signal yielded exact theoretical results, the more general Rician model required an approximation exploiting the generalized gamma (GG) distribution. This was facilitated by an empirical approximation to the GG power parameter to produce a straightforward evaluation of the other model parameters and, subsequently, the JDC after thresholding.

These results will be useful in modeling sonar performance in applications where multiple measurements are subjected to a thresholding operation prior to being combined in a final detection decision.

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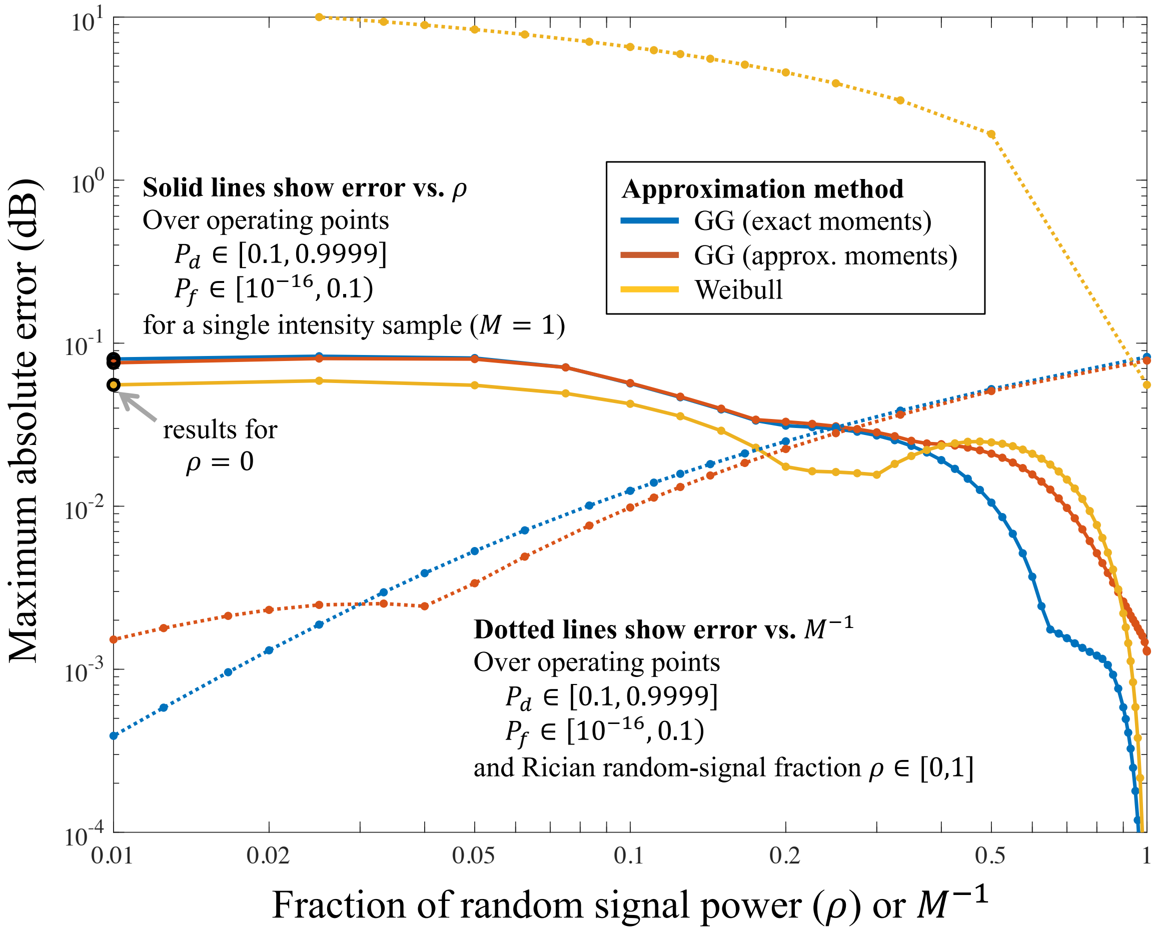

Figure 1: Error in JDC after thresholding for various approximation methods.