TWINKLING IN SONAR SYSTEMS

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Abstract: *Turbulence and other inhomogeneities in the water column cause fluctuations in propagating acoustic signals in the same way that turbulence in the interstellar medium causes stars to twinkle. The amplitude fluctuations bring about signal fading and failure to detect targets well within the theoretical range of the system. Phase fluctuations, however, cause loss of directivity and angular resolution in receiving arrays, spreading of transmitted beams, variations in the apparent arrival direction of signals and fluctuations in their arrival time. Fluctuations also result in an occasional high peak in the signal amplitude, allowing sources or targets to be detected at ranges much greater than predicted by the conventional sonar equation. This paper examines two approaches to predicting the probability of such signal peaks. The first is taken from astronomy and models the underwater medium as a random phase changing screen, or sequence of phase screens separated by a distance equivalent to the width corresponding to the length scale at which the medium remains correlated. The second approach is to apply catastrophe theory to look at how regions of high amplitude move in space. For example, the surface of a swimming pool focuses the light above it to form bright lines on the bottom, known as caustics. These patterns move in time and space following the random variations occurring at the surface. The predictions of these two theories will be compared, and estimates made of the increase in detection range that might be obtained using these occasional high peaks in any realistic scenario.*

Keywords: *Twinkling, Fluctuations, Variability, Scintillation, Phase Screen, Turbulence, Inhomogeneity, Random Medium*

1. INTRODUCTION

Turbulence and other inhomogeneities in the water column cause fluctuations in propagating acoustic signals in the same way that turbulence in the interstellar medium causes stars to twinkle. The amplitude fluctuations bring about signal fading and failure to detect targets well within the theoretical range of the system. Phase fluctuations, however, cause loss of directivity and angular resolution in receiving arrays, spreading of transmitted beams, variations in the apparent arrival direction of signals and fluctuations in their arrival time. Fluctuations also result in an occasional high peak in the signal amplitude, allowing sources or targets to be detected at ranges much greater than predicted by the conventional sonar equation.

Such signal peaks may be of interest to users of sonar systems as they could lead to detection of a target at much greater range than would be suggested by conventional modelling based on spherical (or cylindrical) spreading. This paper examines two approaches to predicting the probability of such signal peaks occurring within a given time period, along with their likely amplitudes.

2. THE ASTRONOMICAL BACKGROUND

There are at least two ways to tackle the problem employed in astronomy. They are the phase screen model and catastrophic optics, and can be briefly explained as follows:

2.1. Phase Screens

Back in the last century, a theory was presented [1], which explained the twinkling of stars as due to layers in interstellar space, or in the earth's upper atmosphere, with moving refractive index spatial structures that refracted the light from distant stars, focusing the light into bright spots that passed across the field of view. These layers were later called phase screens [2].

The fluctuations in perceived intensity of light passing through a single phase screen is relatively easy to model using the distorted wavefront emerging from the phase screen as the initial source [2], then assuming spherical spreading as the light propagates through a uniform medium to the observer and evaluating the effects of interference between light emanating from different parts of the screen. However, for both the starlight case and for underwater acoustics, this is not realistic but is an excessive simplification.

One possible way round the problem is to represent the medium as a succession of elementary slabs of width corresponding to the length scale over which the refractive index fluctuations in the medium, and hence the acoustic fluctuations, remain correlated. Each region is then approximated by a phase-shifting screen at its centre, and these screens are considered to impose independent random phase changes on the propagating wavefronts. See Fig. 1, taken from Uscinsci and Spivack [3], for an explanatory diagram.

The result obtained in this manner represents a single 'look' at the fluctuating medium and its effect on a propagating wavefront; the whole process must be repeated at regular intervals if it is required to represent the effect of a continuously varying medium on an acoustic wave over a period of time.

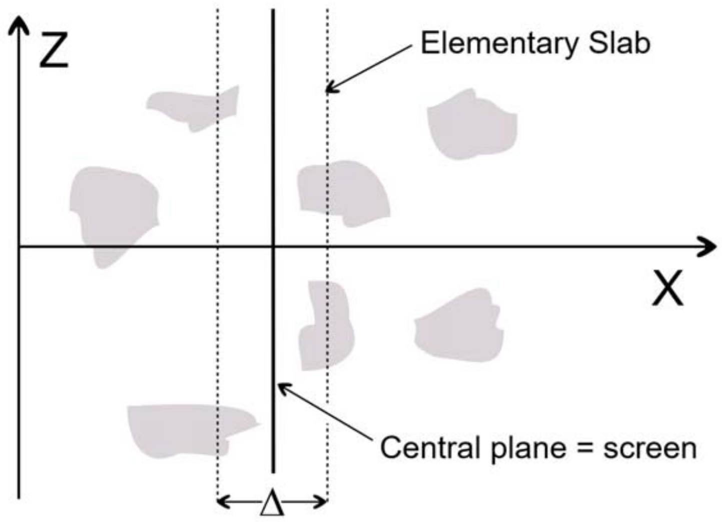

Fig.1: A random medium showing an elementary slab and the central screen onto which the phase of the slab is projected.

2.2. Catastrophic Optics

The phase screen method explained how amplitude at a fixed point in space varies with time (hence 'twinkling'). We can also consider how regions of high amplitude move in space; for example, the surface of a swimming pool focuses the light above it to form bright patterns on the bottom. Another beautiful example is the cusp-shaped curve that sometimes appears on the surface of a cup of coffee (or tea or mug of beer) in bright sunlight; it is caused by the reflection of the sun's rays from the inside of the cup.

In these cases, the regions of high amplitude, or caustics, are structurally stable, which means that they move smoothly as the medium varies in time – they neither pop in and out of existence nor jump from place to place.

Such insights provide the key to mathematical analysis of the caustics: structural stability allows the application of catastrophe theory.

Catastrophe theory was originally developed by Thom [4] and applied to wave propagation by Berry [5]. The theory is derived from topology, the branch of mathematics concerned with the properties of surfaces.

2.2.1. Classification of Caustics

There are seven elementary catastrophes, which describe all possible discontinuities in phenomena controlled by up to four factors. Three examples are shown graphically in Fig. 2 (after Zeeman [6]).

3. CATASTROPHES IN WAVE PROPAGATION

To illustrate the theory, consider the focussing that occurs as a wave propagates through free space from a randomly curved wavefront. This is equivalent to a wave from a

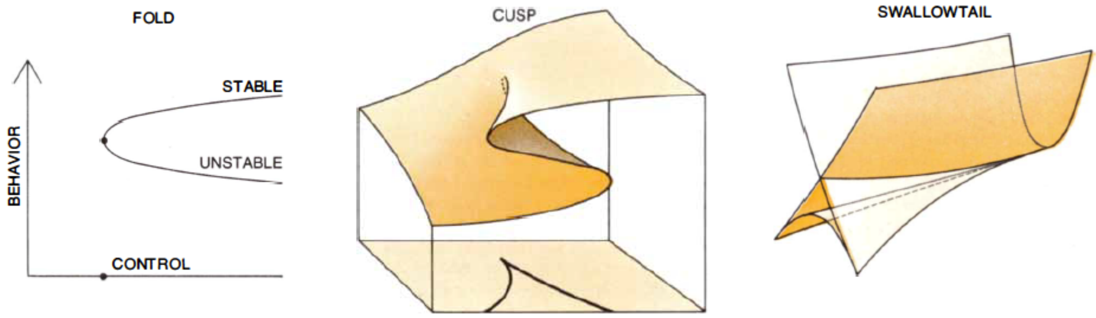

Fig.2: Graphical representation of first three elementary catastrophes, showing surfaces where the first derivative of the control function is equal to zero.

point source moving through a random medium and it can be shown that catastrophe theory applies to this situation [7].

3.1. The Diffraction Integral

So far, we have ignored interference effects: it has been assumed that increased amplitude will be observed where rays intersect. This is true in the particle interpretation of light or when vanishingly small wavelengths are considered, but to accurately model sound and light waves where the wavelength is fixed, interference must be accounted for. To do this, consider the complex wave amplitude $\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{-i\chi(\mathbf{r},t)}$, which encodes the amplitude A and the phase χ. Attention is restricted to waves for which phase can be decomposed as $\chi(\mathbf{r},t) = \Gamma(\mathbf{r}) - \omega t$, with ω the (constant) angular frequency.

Naive attempts to account for interference lead to the semi-classical approximation, where each ray is considered to have an amplitude and phase, and the rays are interfering. However, this approach leads to predictions of infinite amplitude at a caustic, with zero amplitude on one side of it, which conflicts with what is physically possible and with modelling assumptions – for example that □ satisfies a wave equation, so must at least be continuous (see Berry [8]). Indeed, this kind of issue led many mathematicians working in the field of random media to focus on the multiple phase screen approach [9].

The potential usefulness of catastrophe theory was too useful to readily abandon, so the diffraction integral was introduced as a potential fix. For this, the complex wave function is expressed as an integral:

$$\psi(\xi) = C \int_{-\infty}^{\infty} l^{-1/2} e^{ikl(y_0;\xi)} \mathrm{d}y_0 \tag{9}$$

where we have moved to two dimensions for simplicity, taking X to be the coordinate in the main propagation direction (previously Z) and setting $\xi = Y/X$. C is a constant and k is the wavenumber, $2\pi/\lambda$, for wavelength λ.

It is possible to evaluate C [7], but since the wavelength is assumed small, so that k is large compared to l, the integral can be reduced to the approximation

$$\psi(\xi) = R^{-1/2}\left(\frac{k}{2\pi i}\right)^{1/2} e^{ikR} J(\xi) \tag{10}$$

where R is the focal distance (the typical distance at which caustics begin to form) and J is the integral

$$J(\xi) = \int_{-\infty}^{\infty} e^{ik\phi(y_0;\xi)} \mathrm{d}y_0 \tag{11}$$

4. APPLICATION TO SONAR

In the previous Section, we moved between three spatial coordinates (X,Y,Z) and two, dropping the Z. Henceforth, only the two dimensional case will be considered.

As previously discussed, Nye [7] showed that there is a duality between a wave propagating from a randomly curved wavefront in free space, and a wave propagating from a point source in a random medium. Thus, we are able to identify wave amplitude for our (random medium) purposes as being of the form (10), calculated above for a wave in free space, with associated diffraction integral J of the form (11), for φ the standard potential associated with one of the catastrophes.

In the present case, we have just one control dimension - range (or, equivalently, time) so the elementary catastrophe of interest is a fold, with $\phi(y_0;\xi) = \frac{1}{3}\alpha y_0^3 + \xi y_0$. Substituting into the expression for ψ, and, noting that the integrand for J is odd so that we can restrict our attention to the real part of the integral over the positive real numbers. Thus

$$\psi = \psi(\xi) = \left(\frac{2\pi k}{iR}\right)^{1/2} (k\alpha)^{-1/3} \mathrm{e}^{ikR} \mathrm{Ai}(k^{2/3}\alpha^{-1/3}\xi) \tag{12}$$

where the Airy function Ai is defined by $\mathrm{Ai}(s) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3 + st)\mathrm{d}t$

4.1. Parameters for sonar in water

The parameters R and α must now be identified. It is assumed that the values of ε and l_c associated with the random medium are known: ε is defined as rate of dissipation of kinetic energy per unit volume and l_c is a representative scale size for the random medium. More precisely, it may be assumed that the medium is a Gaussian random field $V(r)$ with correlation function $\mathrm{C}(r) = \langle V(r'+r)V(r')\rangle$, where the angle brackets denote an ensemble average. Then l_c is a minimum distance such that C(r) is negligible for $|r| > l_c$ and, for example, if $\mathrm{C}(r) = \mathrm{e}(-|r|^2 / a^2)$, l_c could be taken as 2a

To identify R and α in terms of l_c and ε the aforementioned duality can be exploited [10.] The same diffraction integral can be obtained by considering a wave propagating in free space with wavefront (x_0,y_0) at time t = 0, where

$$x_0 = f(y_0) = \frac{2}{2R}y_0^2 + \frac{1}{3}\alpha y_0^3 \tag{13}$$

The aim is to identify R and α in the free space case in terms of quantities which can be calculated from l_c and ε in the random medium case.

4.1.1. Free space case

Here, R is the focal length, defined as the typical distance from the wavefront to the first caustic (the mean free path)

For α, Metzger *et.al.* [11] use the fact that, writing v_y for fluid velocity in the y direction, there is an extremum of v_y at caustics. This leads to a value of $y = (-2\alpha R^2)^{-1}$ at a caustic, so the typical velocity of rays at a caustic is $v_{typ} = (4\alpha R^2)^{-1}$.

4.1.2. Random medium case

Recall that the medium is assumed to be a Gaussian random field $V(r)$ with correlation function $C(\mathbf{r})$. The crucial point is that the random medium can be approximated as Gaussian white noise σdW_t with parameter $\sigma^2 = v_0^{-3} \int_{-\infty}^{\infty} \partial_{yy} C(x,y)dx$. This approximation, derived by Kulkarny and White [12], leads to a system

$$\frac{1}{v_0}\frac{\mathrm{d}y}{\mathrm{d}x} = v_y, \frac{\mathrm{d}v_y}{\mathrm{d}x} = \sigma \mathrm{d}W_t \tag{14}$$

from which it follows that v_y is a scaled Brownian motion and y is the integral of Brownian motion. Evaluating the stochastic integral, it is found that y is Gaussian, with mean zero and variance $\sigma^2 t^3/3$. Thus, the typical transverse movement of a ray in the random medium at time t scales as $\sigma^2 t^{3/2}$. The assumption is made that caustics occur when a typical ray has deflected by around the correlation length l_c of the medium, so if t_c is the time until caustics appear, then

$$\sigma t_c^{3/2} \approx l_c \tag{15}$$

Next, v_{typ} must be identified in the model. Recall, v_y is Brownian motion, scaled by σ, so estimating a typical value from the standard deviation gives

$$v_{typ} = t^{1/2}\sigma \tag{16}$$

Since σ itself is not a readily measurable quantity, it must be related to the kinetic energy of the random water flow, ε. It is found that [13]

$$\sigma^2 = \varepsilon^2 l_c^{-1} \tag{17}$$

Combining the results above gives

$$R \approx \varepsilon^{-2/3} l_c \qquad v_{typ} \approx \varepsilon^{2/3} \tag{18}$$

4.1.3. Final Amplitude Scaling

Equating the two models, it is found that α scales as $\varepsilon^{2/3} l_c^{-2}$. Recall, from (12), that

$$\psi = \psi(\xi) = \left(\frac{2\pi k}{iR}\right)^{1/2} (k\alpha)^{-1/3} \mathrm{e}^{ikR} \mathrm{Ai}(k^{2/3} \alpha^{-1/3} \xi) \qquad (12)$$

The Airy function Ai is bounded, taking values of modulus less than 1. It is then possible to estimate the intensity I of a sound wave near the caustic as

$$I = (k / R)(k\alpha)^{-2/3} \qquad (19)$$

Substituting the expressions obtained for R and α gives the final result

$$I = \left[(kl_c)^{1/3} \varepsilon^{-2/9}\right] \approx \left[(\lambda / l_c)^{-1/3} \varepsilon^{2/9}\right] \qquad (20)$$

5. IN CONCLUSION

In this paper, the peaks that occur in signals that are fluctuating due to propagation through a turbulent medium have been examined. It has been found that peaks will appear at a typical range R, evaluated in (17) in terms of parameters of the medium, and the intensity, given by (20) in terms of parameters of both the medium and the signal.

This information allows the detection range for a target in a random medium, derived from the modified intensity in (20) to be compared with the range predicted by conventional modelling based on cylindrical or spherical spreading.

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