SIMULATION OF THE TARGET ECHO STRENGTH WITH HIERARCHICAL MATRICES AND THE BOUNDARY ELEMENT METHOD

Boris Dilba[a], Marian Markiewicz[a] , Otto von Estorff[b]

[a]Novicos GmbH, Hamburg, Germany
[b]Institute of Modelling and Computation, Technical University Hamburg, Germany

Boris Dilba, Kasernenstraße 12, 21073 Hamburg, Germany, +4940 300870 50 Fax, dilba@novicos.de

Abstract: *The acoustic scattering properties of underwater objects are of great interest since they characterize their detectable signature, which is also known as Target Echo Strength (TES).*
Depending on the angle of an impinged wave, the scattered response will propagate in all directions with different intensities and thus will result in a varying TES. The Boundary Element Method (BEM) is well suited for TES simulations due to the fact that the solution on domain Ω is determined once the Cauchy Data on the boundary Γ are known. Compared to the Finite Element Method (FEM) the standard BEM has the drawback of fully populated system matrices. Fast Boundary Element Methods like the Fast Multipole Method (FMM) or the Hierarchical Matrix (H-Matrix) approach reduce this drawback effectively. In contrast to the FMM, the H-Matrix procedure generates an approximation of the whole system matrix and provides a low rank arithmetic. The latter enables the generation of a LU-Factorization at costs of $\mathcal{O}(n \log n)$ and provides a direct solution scheme in addition to the iterative GMRES-Method. Since an angular resolution of 0.1° for the monostatic TES leads to 1800 right-hand sides for symmetric objects, the application of the direct solver can be more advantageous than of the iterative one.
In this contribution, different solution strategies for the mono- and bistatic TES calculation by means of the H-Matrix approach will be presented. The differences will be outlined by numerical experiments based on the Benchmark Target Strength Simulation (BeTSSi) submarine model provided by the Forschungsanstalt der Bundeswehr fur Wasserschall und Geophysik (FWG) in Kiel.
Keywords: *BEM, H-Matrix, Target Echo Strength*

1. INTRODUCTION

In the frequency domain the wave propagation is described by the Helmholtz equation

$$\Delta p(\boldsymbol{x}) + k^2 p(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in \Omega \quad k = {}^{\omega}/_{c}\,. \tag{1}$$

The wave number is given by $k = \omega/c$ with the speed of sound c and the angular frequency $\omega = 2\pi f$. Applying integration by parts at the weak formulation of (1) together with the property of the fundamental solution:

$$\begin{gathered} G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} e^{-ik|x-y|}, \\ \Delta G(\boldsymbol{x},\boldsymbol{y}) + k^2 G(\boldsymbol{x},\boldsymbol{y}) = -\delta(\boldsymbol{x},\boldsymbol{y}) \end{gathered} \tag{2}$$

leads to the Boundary Integral Equation (BIE):

$$c(\boldsymbol{y})p(\boldsymbol{y}) + \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})} p(\boldsymbol{x}) \partial\Omega(\boldsymbol{x}) = i\rho\omega \int_{\partial\Omega} G(\boldsymbol{x},\boldsymbol{y}) v_n(\boldsymbol{x}) \partial\Omega(\boldsymbol{x}), \tag{3}$$

where ρ denotes the density of the fluid, p the acoustic pressure and v_n the velocity in normal direction $\vec{n}$. According to the procedure of the collocation method, equation (3) results in the linear system of equations for the discretised Helmholtz equation, cp. [1] and [2],

$$\left(\frac{1}{2} I + K\right) \boldsymbol{p} = i\rho\omega V \boldsymbol{v_n}. \tag{4}$$

For the conventional BEM the matrices K and V are dense and thus have quadratic memory requirements. Due to the fully populated system matrices solving eq. (4) has a complexity of $O(mn^2)$ to $O(n^3)$ with $m < n$. As a result the problem size of the conventional BEM is limited by memory requirements and solution time.

2. HIERARCHICAL MATRIX COMPRESSION

The application of fast compression techniques push the computational limits of the BEM such that larger problems can be solved. An approach to reduce memory requirements and solution time is the concept of hierarchical matrices (H-Matrix), as described in [3] and [4]. The idea of the H-matrix approach is based on the compression of the system matrix. Due to the properties of the fundamental solution (2) and a suitable partitioning of the system matrix, submatrices exist with exponentially decreasing singular values [3], which can efficiently be approximated by a low-rank representation in outer-product form

$$A = \sum_{i=1}^{k} u_i v_i^H. \tag{5}$$

In order to generate a suitable matrix partition the set of degrees of freedom (dofs) $I = \{1, \ldots, n\}$ will geometrically be subdivided into a cluster tree $T(I)$. Starting to subdivide the root $\tau = I$ of the cluster tree $T(I)$ into 2 geometrical separated son clusters $S(\tau) = \{\tau_1, \tau_2\}$ will generate the next level of the cluster tree $T(I)$. The cluster tree $T(I)$ will be generated recursively by applying $S(\cdot)$ to all clusters until a specified minimal

cluster size is reached. Every level of $T(I)$ corresponds to a subdivision of I where the lowest level represents the leaves $\mathcal{L}(T_I)$ of the cluster tree. For the subdivision of the clusters, the corresponding indices will be sorted according to the principal component analysis in order to establish a balanced cluster tree. Based on the cluster trees for the row and column index sets $T(I)$ and $T(J)$ the block cluster tree $T(I \times J)$ is constructed. The admissibility condition

$$\min\{diam(X_\tau), diam(Y_\sigma)\} \leq \eta \, dist(X_\tau, Y_\sigma) \tag{6}$$

with the admissibility factor $0 \leq \eta \leq 1$ states whether a matrix block corresponding to the cluster pairs $\tau \subset T(I)$ and $\sigma \subset T(J)$ is appropriate for a low-rank approximation or not. Here X_τ and Y_σ denote the minimal bounding boxes which include the supports of the test and trial functions of the index sets τ and σ. With eq. (6) the son operation for the block cluster tree $T(I \times J)$ is defined by

$$S_{I\times J}(\tau \times \sigma) = \begin{cases} \emptyset, & if \quad \tau \times \sigma \; fulfills \; (6) \; or \; S_I(\tau) = \emptyset \; or \; S_J(\sigma) = \emptyset \\ S_I(\tau) \times S_J(\sigma), & else. \end{cases} \tag{7}$$

The block cluster tree $T(I \times J)$ is constructed recursively by applying eq. (7) to the roots $\tau = I \subset T(I)$ and $\sigma = J \subset T(J)$ of the corresponding cluster trees. According to eq. (7) for non-admissible blocks 4 sons are generated on every level of the block cluster tree. In order to generate almost quadratic matrix blocks only clusters τ and σ on the same level are considered. The leaves of the block cluster tree define an admissible matrix partition P

$$P \coloneqq \mathcal{L}(T_{I\times J}) \tag{8}$$

of the system matrix which ensures the efficiency of the H-Matrix approximation. The non-admissible blocks of P will be treated the usual way and stored as dense matrix blocks whereas the admissible blocks will be approximated by a low-rank representation (5). In order to efficiently determine low-rank representations of admissible blocks the adaptive cross approximation (ACA) is used. The ACA algorithm finds the rank k of the low-rank representation A_k for a given accuracy ε such that

$$\|A - A_k\|_F < \varepsilon \|A\|_F \tag{9}$$

is fulfilled. Since the stopping criterion is given by the condition

$$\|u_{k+1}\|_2 \|v_{k+1}\|_2 \leq \frac{\varepsilon(1-\eta)}{1+\varepsilon} \|A_k\|_F, \tag{10}$$

the rank of the approximation can be find adaptively. Compared to the FMM, the H-Matrix approach builds approximations of the system matrices K and V of equation (4) and not only of their matrix-vector products. Furthermore, based on the H-Matrix approach an arithmetic can be defined that facilitates operations such as the matrix-vector product, matrix-matrix product and matrix-inversion to be performed with quasi linear complexity. Especially, the hierarchical LU-Decomposition (HLU) of low accuracy can be used as a very efficient preconditioner, which reduces the spectral radius of the system matrix and accelerates the iterative solution process, see [3].

3. REGULARIZATION AND PRECONDITIONING OF THE BIE

The exterior Neumann problem is defined by

$$\begin{aligned} \Delta p(\boldsymbol{x}) + k^2 p &= 0 \quad \boldsymbol{x} \in \Omega \\ \frac{\partial p(\boldsymbol{x})}{\partial n(\boldsymbol{x})} &= f(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial\Omega \\ \frac{\partial p}{\partial r} - ikp &= 0 \;\; r \to \infty. \end{aligned} \tag{11}$$

The corresponding discrete system derived from the boundary integral equation is given by equation (4). It is known that the unique solvability suffers from the occurrence of irregular frequencies at which the system matrix becomes singular, see [5]. In this paper the Burton-Miller (BM) approach is applied in order to ensure a regular system matrix for all real wave numbers k. In this respect the exterior Neumann trace of BIE (3) is taken which leads to the hypersingular BIE

$$c(\boldsymbol{y})\frac{\partial p(\boldsymbol{y})}{\partial n(\boldsymbol{y})} + \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})\partial n(\boldsymbol{x})} p(\boldsymbol{x})\partial\Omega(\boldsymbol{x}) = i\rho\omega \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} v_n(\boldsymbol{x})\partial\Omega(\boldsymbol{x}) \tag{12}$$

and to its discrete form

$$-Dp = i\rho\omega\left(\tfrac{1}{2}I - K'\right)v_n. \tag{13}$$

The combination of BIEs (3) and (12) by a complex coupling parameter α leads to the system of equations

$$\left(\frac{1}{2}I + K - \boldsymbol{\alpha}D\right)p = i\rho\omega\left(V - \boldsymbol{\alpha}\left(\frac{1}{2}I - K'\right)\right)v_n. \tag{14}$$

which is uniquely solvable for all real wavenumbers k, if $\Im(\alpha) \neq 0$. According to Kress [6], a good choice of α with respect to a minimal condition number is given by $\alpha = i/k$.

However, derived from the Calderón identities an optimal choice for α is the Neumann-to-Dirichlet (NtD) map, since the system matrix

$$\left(\frac{1}{2}I + K - \boldsymbol{NtD}D\right)p = Ip. \tag{15}$$

becomes identity. Choosing the NtD operator for α rather than a complex scalar improves the eigenvalue clustering of the system matrix and regularizes the problem at the same time. Hence, the NtD coupling operator is a good preconditioner of eq. (14). In that respect, the complex valued coupling parameter $\alpha = i/k$ can be understood as the local application of the Sommerfield radiation condition on the surface $\partial\Omega$ and thus as simple approximation of the NtD operator. More accurate approximations of the NtD map are given by the On-Surface Radiation Condition (OSRC) method [7]. The idea of OSRC is based on finding a local surface approximation of the NtD operator. A good choice for NtD is given by the non-local square root operator

$$NtD \approx \frac{1}{ik}\left(1 + \frac{\Delta_{\Gamma}}{k_{\varepsilon}^2}\right)^{-1/2}, \tag{16}$$

where Δ_{Γ} denotes the surface Laplace-Beltrami operator [7]. The localization of (16) is achieved with a Pade expansion of size Np, such that the application of the NtD approximation can be reduced to solving a set of (Np+1) sparse linear systems of equations. In addition to the analytic OSRC preconditioning, the H-Matrix compression provides an algebraic approach in order to reduce the number of iteration steps. Performing a LU factorization of the system matrix (14) with approximated low-rank Matrix-Matrix operations leads to

$$A \approx L_{\delta} U_{\delta}. \tag{17}$$

By defining the rounding precision delta for these operations, hierarchical LU factorizations of arbitrary accuracy can be achieved. For preconditioning of the regular BM matrix (14) with $\alpha = i/k$, a low accuracy of $\delta = 10^{-1}$ is sufficient to effectively reduce the number of GMRES steps.

4. TARGET ECHO STRENGTH SIMULATION

The TES is defined by the ratio of the pressure amplitudes between incident wave (p_0) and the scattered pressure (p_s) at the receiver location (r)

$$TES = 20 \cdot log_{10}\left(\frac{|r - r_0||p_s|}{|p_0|}\right) \tag{18}$$

scaled by the distance between the object (r_0) and (r). For the monostatic TES source and receiver locations coincide such that every sample point in the far field yields an additional right-hand side for the system of equations (14). A fine elevation-angle resolution of 0.1° between 0° and 180° requires 1800 system solutions of (14) and thus leads to a high computational effort. In contrast, different locations for source and receiver points are considered for the bistatic TES. For every impinging wave multiple far field evaluations are performed according to sampling resolution. In general, less sources are considered for the bistatic TES such that, the number of required system solutions and thus the computational effort is smaller compared to the monostatic case.
In order to assess the efficiency of the aforementioned preconditioning strategies with respect to the different TES calculations, we consider the hard-walled scattering problem of the submarine model from the BeTSSi workshop [8]. Due to the symmetric surface of the submarine the regularised system matrix (14) is block symmetric according to

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}. \tag{19}$$

In addition to the standard ($\alpha = i/k$) BM approach, its H-LU preconditioned variant and the OSRC regularization, a block Jacobi accelerated version of the standard BM is also considered. Exploiting the symmetric structure (19), the Jacobi block preconditioner

$$P_{0.1}^{-1} = \begin{pmatrix} U_1^{-1}L_1^{-1} & 0 \\ 0 & U_1^{-1}L_1^{-1} \end{pmatrix} \tag{20}$$

is determined by a low accuracy ($\delta = 10^{-1}$) H-LU of the diagonal block $A_1 = L_1 U_1$.
The numerical experiments are performed with $c = 1500\,m/s$ and $\rho = 1000\,kg/m^3$ at 1kHz, 3kHz and 5kHz. The corresponding surface discretizations with 10 elements per wave

length and constant shape functions for the field values lead to linear systems with 115.738, 893.140 and 1.980.562 unknowns. Since only the BM-HLU approach requires the assembly of the full H-matrix, its set up costs are higher than for the other preconditioning strategies, as illustrated in figure 1.

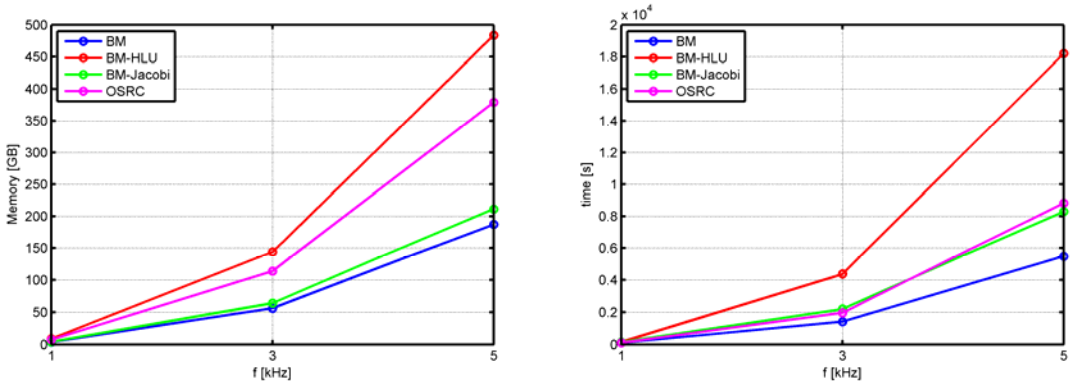

Fig.1: Memory requirements (left) and assembly times (right)

The lowest memory and time consumption is observed for the standard BM approach, which exploits the block symmetric structure of the system matrix. In order to efficiently apply the OSRC regularization operator, the K and D matrices have to be kept separate which leads to twice the assembly costs of the BM approach. In contrast, the additional costs to set up the block Jacobi preconditioner are only slightly higher than for the BM variant. The efficiency of the preconditioning strategies for the different frequencies is shown in figure 2. Without improving the eigenvalue clustering of the system matrix the standard BM variant requires the largest number of GMRES iterations in order to determine a solution for a relative error of $\varepsilon = 10^{-6}$. Even though the OSRC method requires only half of the iteration steps its solution time is about the same as without preconditioning. Due to the separate handling of K and D the reduced number of iteration steps is compensated by twice as high matrix-vector-product costs. A more effective reduction of the solution costs is observed for the block Jacobi variant. Compared to standard BM method approximating the inverse of the block diagonal reduces the number of GMRES steps by a factor of 3 while the solution time is halved. However, the smallest number of steps and the fastest solution time are achieved by the low accuracy ($\delta = 10^{-1}$) H-LU preconditioning of the full H-Matrix.

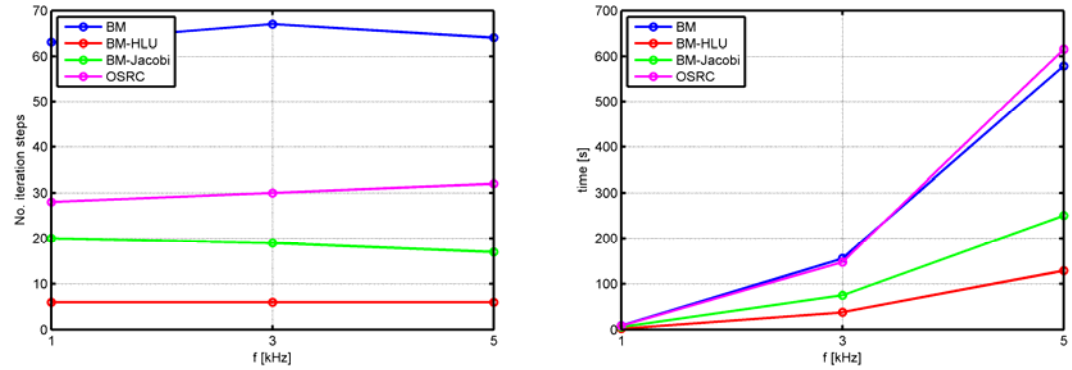

Fig.2: No. of iteration steps (left) and required solution time for 1 rhs (right)

A comparison of the total computational (assembly and solution) times for determining the bistatic and monostatic TES is given in figure 3. Due to the much lower solution effort the overall times for assessing the bistatic TES reflect the assembly costs of the H-matrices.
For the monostatic TES calculation the opposite is the case. The number of 1800 right-hand sides increases the influence of the solution costs such that the impact of the matrix assembly effort is lowered.

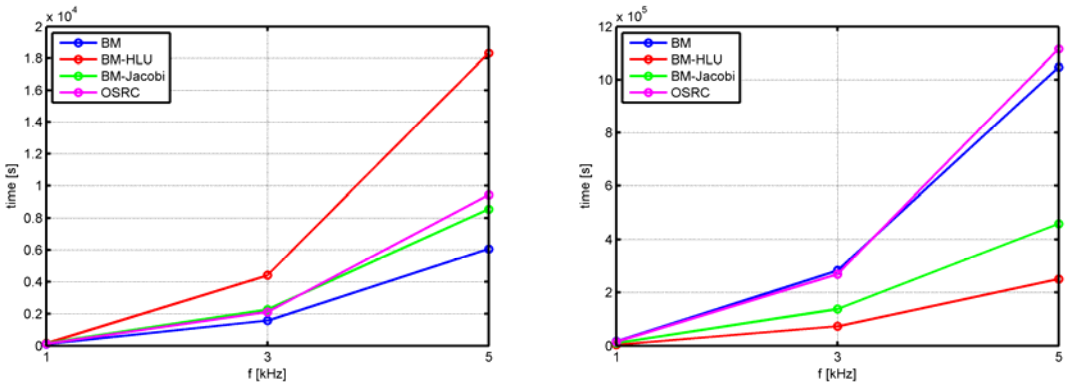

Fig.3: Total time for determining the bistatic TES (left) and the monostatic TES (right)

5. CONCLUSIONS

The presented preconditioning strategies can effectively reduce the computational costs for the TES simulation. For the H-LU based preconditioners already a low accuracy of $\delta = 10^{-1}$ is sufficient to significantly minimize the number of GMRES steps. By exploiting the block symmetric structure of the system matrix both the assembly time and the memory requirements can be reduced by 50%. Furthermore, it enables an efficient set up of H-LU based block Jacobi preconditioners of low accuracy which is shown to effectively reduce the number of iteration steps. Although choosing the NtD OSRC approximation as BM regularization operator improves the convergence rate of the GMRES method, the solution time suffers from doubled MVP costs, and hence its efficient application as a preconditioner is limited to the bistatic TES simulation. In order to reduce the overall time for TES simulations the preconditioner should be chosen with respect to the number of rhs. The more system solutions are needed the less the solution time per rhs should be.

REFERENCES

[1] **von Estorff, O.**, Boundary Elements in acoustics- Advances and Applications, WIT Press, Southampton, 2000

[2] **Wu, T.W.**, Boundary Element Acoustics: Fundamentals and Computers Codes, WIT Press, 2000

[3] **Bebendorf, M.**, Hierarchical Matrices: A Means to Efficiently Solve Elliptic Boundary Value Problems, Volume 63 of Lecture Notes in Computational Science and Engineering (LNCSE), Springer-Verlag, 2008

[4] **Hackbusch, W.**, Hierarchische Matrizen: Algorithmen und Analysis, Springer-Verlag, 2009

[5] **Burton A. J., Miller G. F.**, The application of integral equation methods to the numerical solution of some exterior boundary value problems, Proc. Roy. Sot. London Ser. A 323, 1971

[6] **Kress R.**, Minimizing the condition number of boundary integral operators in acoustics and electromagnetic scattering, The Quarterly Journal of Mechanics and Applied Mathematics, 38(2), 1985

[7] **Antoine X., Darbas M.**, Alternative integral equations for the iterative solution of acoustic scattering problems, The Quarterly Journal of Mechanics and Applied Mathematics, 58(1), 2005
[8] **Fiedler C., Schneider H.G.**, BeTSSi-Sub – Benchmark Target Strength Simulation Submarine, Technical Report, Forschungsanstalt der Bundeswehr für Wasserschall und Geophysik, Kiel, 2002