WHISPERING GALLERY WAVES IN HORIZONTAL PLANE NEAR THE CURVED ISOBATHS IN SHALLOW WATER

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Abstract: *Interference pattern of the sound field from the point source in horizontal plane in shallow water waveguide in vicinity of curvilinear isobath is studied. This situation can take place in area of the coastal wedge with curvilinear apex (a lagoon or part of a lake). It is shown that formation of waves (modes) of whispering gallery is possible for the sound source which is close enough to the coastal line. Properties of these waves are studied analytically and using numerical modelling and WKB approximation.*

Keywords: *sound in shallow water, whispering gallery waves, horizontal refraction*

1. INTRODUCTION

Interference structure of the sound field in horizontal plane in shallow water with parameters variable in horizontal plane is studied within the framework of the so-called 3D problem. Formally this problem can be solved using separation of the vertical coordinate (depth) in the wave equation assuming that the depth dependence is provided by adiabatic waveguide modes. The modal amplitudes satisfy a two-dimensional dispersive wave equation which can be solved by different methods (PE, ray approximation, modal decomposition). In the paper, it is shown that in a shallow water waveguide with curvilinear isobaths (lagoon, lake) there exist specific solutions of this equation, concentrated approximately along isobaths (whispering gallery waves [1]). These solutions are, by their very nature, waveguide modes in the horizontal plane.

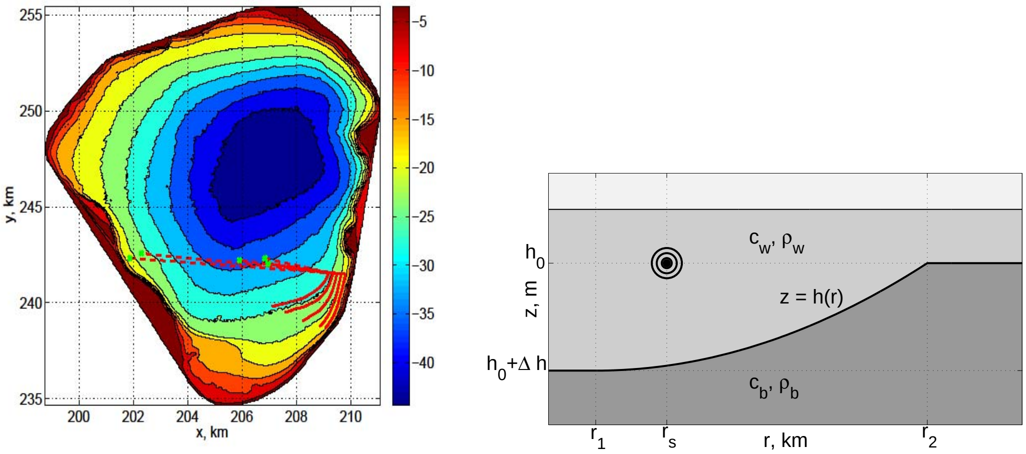

Fig.1: Bathymetry of Lake Kinneret (left) and radial section of a cylindrically-symmetric lake (right) by a vertical plane.

2. PROBLEM DESCRIPTION

We consider a simplified model of an open locally cylindrically-symmetric bay that will be used to draw simple analytical estimates related for the whispering gallery formation. Consider a cylindrical coordinate system (r,θ,z), where z denotes depth. The bay has local rotational symmetry about z axis, and the plane $z=0$ is its surface. The bottom relief is described by the function $z=h(r)$ (i.e. depth does not depend on the azimuth θ) inside certain sector $\theta\in[\theta_{\min},\theta_{\max}]$. We restrict ourselves to the case when h is constant outside a finite interval $[r_1,r_2]$, while within this interval it is a quadratic polynomial:

$$h(r)=\begin{cases} h_1,\ if\ \ r<r_1, \\ ur^2+vr+w,\ if\ \ r_1\le r\le r_2, \\ h_2,\ if\ \ r>r_2. \end{cases} \qquad (1)$$

The coefficients u,v,w are chosen in such a way that function $h(r)$ is continuous. A vertical section of the lake with the bottom relief described by the function (1) is shown in Fig. 1. We assume that outside the sector $\theta\in[\theta_{\min},\theta_{\max}]$ the bottom relief is such that the sound wave can leave it freely. The sound speed and density in the water column

$0 \le z \le h(r)$ are c_w and ρ_w respectively, while the same parameters in the bottom are c_b and ρ_b.

Assume that a time-harmonic point source of frequency f is located at $r = r_s$, $z = z_s$ and $\theta = \theta_s = 0$. Our goal is to compute the sound pressure field $P(x, y, z)$ formed by the source in this water column and to predict the conditions under which a whispering gallery wave can be formed.

Although the model is highly idealized, the main propagation features (according to the argument of Babich and Buldyrev [1]) can be also observed in the case of the arbitrary concave boundary.

3. HORIZONTAL REFRACTION EQUATIONS

The solution of the propagation problem described in the previous section can be obtained in the form of the so-called modal decomposition [2,3]

$$P = \sum_{j=1}^{N_m} A_j(r,\theta)\phi_j(r,z), \tag{1}$$

where the modal functions $\phi_j(r,z)$ are obtained from the solution of the acoustical spectral problem [2]:

$$\begin{aligned} &\phi_{zz} + \frac{\omega^2}{c^2}\phi - k^2\phi = 0 \\ &\phi(0) = 0, \qquad \phi|_{z\to\infty} \to 0 \\ &\phi\big|_{z=h^+} = \phi\big|_{z=h^-}, \qquad \left.\frac{\phi_z}{\rho}\right|_{z=h^+} = \left.\frac{\phi_z}{\rho}\right|_{z=h^-}. \end{aligned} \tag{2}$$

The squared horizontal wavenumber k^2 plays the role of a spectral parameter in (2). Consider the eigenvalues of (6) $k_1^2 > k_2^2 > \ldots > k_{N_m}^2$ whose eigenfunctions $\phi_1(z), \phi_2(z), \ldots$ have maxima inside the water column and decay at $z \to \infty$ (the latter are called trapped modes). It is clear that the far field the will be dominated by the trapped modes, while continuous spectrum modes can be neglected. Note that in principle the interface depth h in the formulation of (2) depends on r. Hence the mode wavenumbers $k_j = k_j(r)$ and the mode functions $\phi_j = \phi_j(r,z)$ also parametrically depend on r.

It can be shown that the mode amplitudes $A_j(r,\theta)$ satisfy the so-called horizontal refraction equations [2,3]. In the case when the mode coupling effects are negligible (e.g. for very slow variation of $h(r)$), these equations can be written as

$$\frac{1}{r}\left(rA_{j,r}\right)_r + \frac{1}{r^2}A_{j,\theta\theta} + k_j^2 A_j = -\frac{\delta(r-r_s)\delta(\theta-\theta_s)\phi_j(z_s)}{r\rho(z_s)}. \tag{3}$$

The Helmholtz-type equations (3) are called *adiabatic* horizontal refraction equations (HREs). Note that equation (3) resembles the 2D Helmholtz equation in a rotationally-invariant bounded domain for which the usual theory of whispering gallery waves is developed [1].

4. RAY-THEORETICAL CONDITIONS FOR WHISPERING GALLERY FORMATION

In this section we derive a simple ray-theoretical inequality for the media parameters and the waveguide geometry that ensures the formation of the whispering gallery waves. We start with the Snell's law in the cylindrical coordinate system

$$\frac{v_j^{\mathrm{p}}(r)}{r\sin(\chi(r))} = \text{const.} \tag{4}$$

The horizontal phase velocity v_j^{p} j-th mode in (4) of is defined by the formula $v_j^{\mathrm{p}}(r) = \omega/k_j(r)$, and the $\chi(r)$ is the incidence angle (i.e. the angle between the ray and the radius vector at a given point).

Assume that some ray radiated by the source located at $r = r_s$ is trapped inside a ring $r_{\min} < r < r_{\max}$ and therefore contributes to the formation of a whispering gallery. If $r_{\min}$ and $r_{\max}$ are the turning points of this ray, then it is obvious that $\chi(r_{\min}) = \chi(r_{\max}) = \pi/2$. Furthermore, from (8) we immediately obtain

$$\frac{v_j^{\mathrm{p}}(r_{\min})}{r_{\min}} = \frac{v_j^{\mathrm{p}}(r_s)}{r_s\sin(\chi(r_s))} = \frac{v_j^{\mathrm{p}}(r_{\max})}{r_{\max}}. \tag{5}$$

This equality can be recast as

$$\frac{v_j^{\mathrm{p}}(r_{\min})}{r_{\min}} > \frac{v_j^{\mathrm{p}}(r_s)}{r_s}, \qquad \frac{v_j^{\mathrm{p}}(r_s)}{r_s} < \frac{v_j^{\mathrm{p}}(r_{\max})}{r_{\max}}. \tag{6}$$

Thus we conclude that the whispering gallery can be formed by a certain family of rays under condition that the function

$$\zeta(r) = \frac{v_j^{\mathrm{p}}(r)}{r} = \frac{\omega}{k_j(r)r} \tag{7}$$

has a local minimum on the interval $[r_1, r_2]$. In principle we can drop the first inequality in (10) as it is always fulfilled for $h(r)$ given by (1). It is also clear that the whispering gallery wave is formed by the family of rays emitted from the source at such angles $\chi(r_s)$ that

$$\sin(\chi(r_s)) > \frac{r_{\max}}{r_s}\frac{v_j^{\mathrm{p}}(r_s)}{v_j^{\mathrm{p}}(r_{\max})}.$$

Note that the condition (6) takes the media parameters (such as sound speeds, densities, depth, etc) into account through the horizontal wavenumber $k_j(r)$ (and hence through the phase velocities $v_j^{\mathrm{p}}(r)$).

It is also clear from (6) that the solutions of (3) localized near the minimum $r = r_0$ of the function (7) exist even when the respective isobath is just an arc of a circle (and not a closed curve as in [1]). Such isobaths are ubiquitous in the neighborhood of a coast line of any convex bay. In the following section we develop a WKB theory for the whispering gallery modes localized near such isobaths and study an interference pattern of formed by these modes.

5. WHISPERING GALLERY MODES IN AN OPEN BAY

In this section we study the modal structure of the whispering gallery waves field in an open bay. For simplicity we approximate the non-decreasing function $k_j = k_j(r)$ by a step function

$$k_j(r) = \begin{cases} k_j, \, if \;\; r < r_2, \\ \hat{k}_j, \, if \;\; r > r_2. \end{cases}$$

In this case the condition (6) is clearly satisfied, and $r_{\max} = r_2$. It is also obvious that our waveguide for the whispering gallery waves is formed by the circle $r = r_2$ and the caustic $r = r_{\min}$ in the sector $\theta \in [\theta_{\min}, \theta_{\max}]$ with open boundaries $\theta = \theta_{\min}$ and $\theta = \theta_{\max}$. It resembles the Pekeris-type waveguide in a shallow sea, although by contrast to the latter it is formed in a horizontal plane. In our simple model the whispering gallery modes are clearly of "reflected-refracted" type, while in reality (when $k_j(r)$ varies continuously) each mode has two turning points at $r_{\min}$ and $r_{\max}$ and therefore belongs to "refracted-refracted" type (which is more similar to the trapped modes of a deep-sea underwater sound channel).

Consider homogeneous HRE corresponding to (3):

$$\Delta A_j + (k_j(r))^2 A_j = 0. \tag{8}$$

Clearly, it can be solved by the separation of variables, and the modal amplitudes $A_j(r,\theta)$ can be represented as superpositions of waves

$$A^{\pm}_{j\nu}(r,\theta) = Q^{\pm}_{j\nu}(r)\mathrm{e}^{\pm \mathrm{i}\nu\theta}, \tag{9}$$

Where signs $\pm$ correspond to the waves propagating in positive and negative directions of θ respectively. If the source is located at $\theta = 0$ the assumption that the waves can leave the sector through the boundaries $\theta = \theta_{\min}$ and $\theta = \theta_{\max}$ is equivalent to the following statement. For $\theta > 0$ the solution of (3) is comprised by the functions of the form (9) with the "positive" exponentials $\mathrm{e}^{\nu\theta}$ (i.e. the waves propagating counterclockwise), while for $\theta < 0$ the solution consists of the waves (9) with "negative" exponentials $\mathrm{e}^{-\nu\theta}$ (the latter waves propagate clockwise). Hereafter we consider only the solution for $\theta > 0$ retaining only the waves of the form $Q_\nu \mathrm{e}^{\mathrm{i}\nu\theta}$ (they are called horizontal modes). In order to simplify the notation we will omit the index j (thus we fix certain mode number in the decomposition (1)).

Substituting ansatz $Q_\nu(\mathrm{r})\mathrm{e}^{\mathrm{i}\nu\theta}$ into the HRE (8), we obtain the Bessel-type equation with variable coefficient $k(r)$ for the function $Q_\nu(r)$:

$$Q_{\nu,rr} + \frac{1}{r}Q_{\nu,r} + \left((k(r))^2 - \frac{\nu^2}{r^2}\right)Q_\nu = 0. \tag{10}$$

Its solution must be bounded at $r = 0$, while at $r \to \infty$ it has to satisfy the radiation condition. Our goal is to obtain the solutions of (10) localized inside the ring $r_{\min} < r < r_{\max}$. The solutions $Q_\nu(\mathrm{r})\mathrm{e}^{\mathrm{i}\nu\theta}$ of this kind are called whispering gallery modes, and they are trapped between the penetrable boundary $r_{\max} = r_2$ and the caustic at $r = r_{\min}$.

6. WKB ESTIMATES OF WHISPERING GALLERY AZIMUTAL WAVENUMBERS

Boundary-value problem for Eq. (10) can be considered as an eigenvalue problem with ν^2 as a spectral parameter. In this section we compute the solutions of this eigenvalue problem corresponding to the whispering gallery modes using the WKB-approximation for the step function $k(r)$ (this can be done analytically).

Following the usual scheme of the WKB method [2], we substitute the ansatz $Q = B(r)\mathrm{e}^{\mathrm{i}\Phi(r)}$ into Eq. (10). The real part of resulting equality has the form

$$B_{rr} + \frac{1}{r}B_r + \left((k(r))^2 - \frac{\nu^2}{r^2}\right)B - (\Phi_r)^2 B = 0.$$

Assuming that the amplitude $B(r)$ varies slowly we neglect first two terms. The a first-order ordinary differential equation is obtained for the phase functions $\Phi(r)$:

$$\Phi_r = \pm\sqrt{(k(r))^2 - \frac{\nu^2}{r^2}}.$$

The solution of this equation is trivial, and we find

$$\Phi(r) = \int_{r_{\min}}^{r_{\max}} \sqrt{(k(r))^2 - \frac{\nu^2}{r^2}}dr. \tag{11}$$

If the wavenumber $k(r)$ is constant for $r_{\min} < r < r_{\max}$, then the integral in (11) can be evaluated explicitly:

$$\Phi(r) = \nu\left(\pm\sqrt{\frac{(k(r))^2 r^2}{\nu^2} - 1} - 2\arctan\left(\frac{k(r)r}{\nu}\right) \pm \sqrt{\frac{(k(r))^2 r^2}{\nu^2} - 1}\right)\Bigg|_{r_{\min}}^{r_{\max}}, \tag{12}$$

where the signs $\pm$ correspond to the signs in the expression for Φ_r.

Note that from the imaginary part of the equation obtained by introducing the WKN ansatz to (10) the following expression for the amplitude $B(r)$ can be easily obtained:

$$B(r) = \frac{C}{\sqrt[4]{k^2 - \frac{\nu^2}{r^2}}}.$$

Let us now compute the phase increment that corresponds to the full cycle of the ray travelling from the turning point $r = r_{\min}$ at the caustic to the outer boundary $r_{\max} = r_2$ and back. Consider a whispering gallery mode that corresponds to the ray that with the grazing angle α at the boundary $r = r_2$. Within the WKB-approximation it can be written as

$$Q_\nu(r)\mathrm{e}^{\nu\theta} \approx B_\nu(r)\mathrm{e}^{\mathrm{i}\nu\theta \pm \mathrm{i}\int\sqrt{k^2 - \frac{\nu^2}{r^2}}dr}. \tag{13}$$

Thus, the tangential component of the wavenumber is ν/r_2, while its normal (or radial) component reads as $\sqrt{k^2 - \nu^2/r_2^2}$. Obviously, the grazing angle α at $r = r_2$ corresponding to this mode can be obtained from the equality

$$\cos\alpha = \frac{\nu}{kr_2}. \tag{14}$$

This equality allows us to make a convenient change of variables and quantize the angle α instead of the azimuthal wavenumber ν. Note that the radial coordinate of the inner turning point of the ray can be computed as $r_{\min} = \nu/k$.

The whispering gallery waves undergo the total internal reflection at $r = r_2$, hence the usual condition must be fulfilled (we introduced the horizontal refractive index $\hat{k}/k = n$):

$$\alpha < \arccos\left(\frac{\hat{k}}{k}\right) = \arccos(n). \qquad (15)$$

The phase increment for the WKB-wave propagation from $r_{\min}$ to $r_{\max}$ and back is

$$\Delta\Phi_p = 2\int_{\frac{\nu}{k}}^{r_2} \sqrt{k^2 - \frac{\nu^2}{r^2}}dr = 2\nu\left(\sqrt{\frac{k^2 r_2^2}{\nu^2} - 1} - 2\arctan\left(\frac{kr_2}{\nu} + \sqrt{\frac{k^2 r_2^2}{\nu^2} - 1}\right) + 2\arctan(1)\right). \qquad (16)$$

At $r = r_2$ the phase increment due to the total internal reflection amounts to $\Delta\Phi_r = \arg(R)$, where R is the reflection coefficient (see e.g. [2]). At $r = r_{\min} = \nu/k$ the wave if reflected from the caustic, and a phase jump of $\Delta\Phi_c = -\pi/2$ occurs [1]. Thus computing the full-cycle phase increment we can write the WKB quantization condition as

$$\Delta\Phi_p + \Delta\Phi_c + \Delta\Phi_r = 2\pi m.$$

Writing this condition in terms of grazing angle α (instead of ν), we obtain

$$2kr_2\cos\alpha\left(\tan\alpha - 2\arctan\left(\frac{1}{\cos\alpha} - \tan\alpha\right) + 2\cdot\frac{\pi}{4}\right) - \frac{\pi}{2} - 2\arccos\left(\frac{\sin\alpha}{\sqrt{1-n_2^2}}\right) = 2\pi m.$$

For small α an approximate solution of this equation reads as

$$\alpha_m = -2\sqrt{\frac{s}{3}}\sinh\left(\frac{1}{3}\operatorname{arcsinh}\left(\frac{3t}{2s}\sqrt{\frac{3}{s}}\right)\right), \qquad (17)$$

where

$$s = \frac{6(1-n^2)}{2kr_2(1-n^2)^{3/2} + n^2}, \quad \text{and} \quad t = \frac{-3\pi(m+3/4)}{kr_2\left(1 + \frac{n^2}{2kr_2(1-n^2)^{3/2}}\right)}.$$

Note that the angle α_m increases with the increasing of *m*, hence there is only a finite number of values satisfying $\alpha_m < \arccos(n)$. Thus, for every set of parameters waveguide admits only finite number of whispering gallery modes. The values of grazing angle can be found by (17), while the respective values of ν can be obtained from Eq. (14).

Now we present a numerical example, where formula (17) is used for the evaluation of α_m and ν, and then the modes $Q_\nu(\mathrm{r})\mathrm{e}^{\mathrm{i}\nu\theta}$ are computed by solving (10). The acoustic filed produced by a point source (see (3)) near the arc $r = r_2$ is subsequently obtained by the as the whispering gallery modes superposition. We set the following values for sound speed and density in the water and in the bottom: $c_w = 1500$ m/s, $\rho_w = 1$ g/cm^3, $c_b = 2000$ m/s, $\rho_b = 1.8$ g/cm^3. The bottom depth is $h_1 = 26$ m for $r < r_2$, $h_2 = 24$ m for $r > r_2$, while the radius r_2 equals to 6 km. The radial components $Q_\nu(\mathrm{r})$ of the whispering gallery modes for the source frequency of 120 Hz are shown in the left panel of Fig. 2. The acoustic field due to a point source can be computed as a superposition of such modes (right panel in Fig. 2).

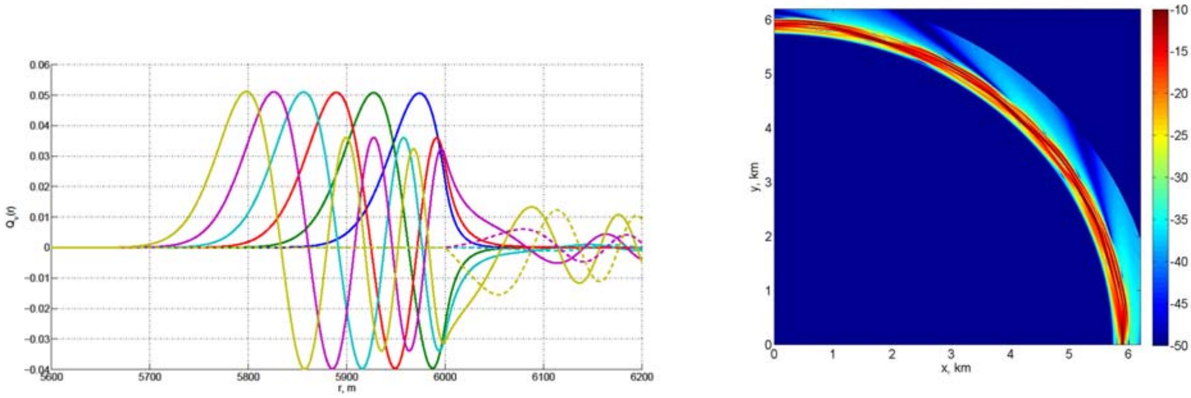

Fig.2: A radial functions $Q_\nu(\mathrm{r})$ *for the whispering gallery modes in our example (left) and the interference pattern of whispering gallery modes near* $r = r_2 = 6\,\mathrm{km}$ *(right).*

7. CONCLUSION

In this study it is shown that in an open shallow-water bay or lake with variable bathymetry a waveguide where the whispering gallery modes are excited may exist in the horizontal plane. This waveguide can be formed in vicinity of curvilinear isobaths and the number of modes depends on frequency and radius of curvature. Remark that this whispering gallery modes can exist in real conditions, for example for the frequency of few hundreds of herz and radius of curvature of about 5 km.

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