# WATER COLUMN SOUND SPEED PROFILE RECOVERY VIA ADJOINT MODELLING*

Eftychia Karasmani[a] and John S. Papadakis[a,b]

[a]Institute of Applied and Computational Mathematics, Foundation for Research and Technology-Hellas, P.O. Box 1385, 71110 Heraklion, Crete, Greece.
[b]Department of Mathematics and Applied Mathematics, University of Crete, P.O. Box 2208, 71409 Heraklion, Crete, Greece.

**Abstract:** In this work, we exploit the adjoint optimal control method developed for geoacoustic inversions by J. S. Papadakis et al., for recovering the sound speed profile (SSP) in the water column. In particular, the direct propagation problem is modelled via the wide angle parabolic approximation (WAPE) and the cost function quantifies the discrepancy between the observed acoustic field and the field predicted by the model. A gradient descent iterative scheme is used for the minimization of the cost function. Here we assume that the actual SSP can be written as a perturbation of a piecewise linear background profile. The perturbation is a linear combination of the empirical orthogonal functions (EOFs) which are coming from statistical analysis of long term observations (historical data) and depend on the location. We calculate analytically and compute numerically the derivatives of the cost function with respect to all the unknown parameters: the three sound speeds that characterize the piecewise linear background profile, the depth of the minimum sound speed and the coefficients which determine the amplitude of the basis functions. Inversions are performed for simulated data.

***Keywords:*** *adjoint optimal control method, parabolic approximation, sound speed profile, water column, empirical orthogonal functions.*

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## INTRODUCTION

In this work, we exploit the adjoint optimal control method developed for geoacoustic inversions in [1] and [2], for recovering the sound speed profile (SSP) in the water column. In particular, the direct propagation problem is modelled via the wide angle parabolic approximation (WAPE), see [3], and the cost function quantifies the discrepancy between the observed acoustic field and the field predicted by the model. A gradient descent iterative scheme is used for the minimization of the cost function. Here we assume that the actual SSP can be written as a perturbation of a piecewise linear background profile. The perturbation is a linear combination of the empirical orthogonal functions (EOFs) which are coming from statistical analysis of long term observations (historical data) and depend on the location. We calculate analytically and compute numerically the derivatives of the cost function with respect to all the unknown parameters: the three sound speeds that characterize the piecewise linear background profile, the depth of the minimum sound speed and the coefficients which determine the amplitude of the basis functions. To assess the performance of the method, two test cases are exhibited. In the first test case an artificial example is proposed, where we use an unperturbed piecewise linear SSP. In the second test case we consider five different perturbed SSPs, using the EOFs, see [4]. Inversions are performed for simulated data. The estimations obtained are compared with the actual values of the parameters.

## 1. FORMULATION OF THE INVERSE PROBLEM

We consider a two dimensional (azimuthal symmetric) sea environment which consists of a water column of depth $z_B$, constant density $\rho_w$, depth dependent sound speed $c(z)$, and a semi-infinite bottom of constant density $\rho_B$ and sound speed $c_B$. A point harmonic source is located at a depth $z_s$ inside the water column, see Fig. 1.

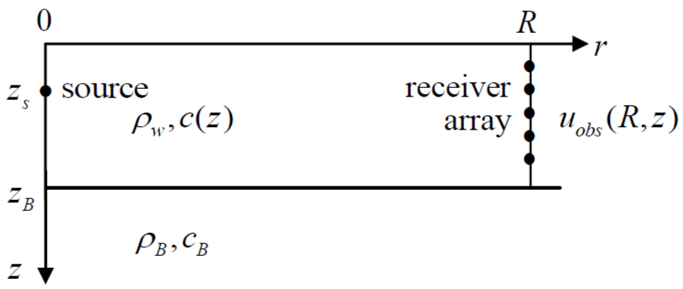


*Fig.1: Schematic for the inverse problem.*

In the sequel we assume that the sound speed $c(z)$ can be written as a linear combination of some basis functions $f_j$ as follows

$$c(z) = c_0(z) + \sum_{j=1}^{N} a_j f_j(z) \text{ for j} = 1, 2, \ldots, N,$$

where the coefficients $a_j$ are the amplitudes of $f_j$ and $c_0(z)$ is a piecewise linear background sound speed profile given by

$$c_0 = c_{0,1} + (c_{0,2} - c_{0,1})H(z - z_m),$$

with $H$ being the Heaviside function, $z_m$ the depth of the minimum sound speed and $c_{0,1}, c_{0,2}$ as follows

$$c_{0,1}(z) = c_1 + \frac{c_2 - c_1}{z_m} z,\ 0 \le z \le z_m$$

$$c_{0,2}(z) = c_2 + \frac{c_3 - c_2}{z_B - z_m}(z - z_m),\ z_m \le z \le z_B.$$

Given the observed acoustic field $u_{obs}$ on the vertical receiver array at a distance $R$ from the source, we want to estimate $c_0(z)$ and $a_i$, assuming that $f_i$ are known.

## 2. THE FORWARD MODEL

The WAPE boundary value problem is formed as follows:

$$Lu = 2ik_0\left[1 + \frac{1}{4}(n^2 - 1)\right]u_r + u_{zz} + k_0{}^2(n^2 - 1)u + \frac{i}{2k_0}u_{rzz} = 0$$

$$u(0, z) = S(z, z_s)$$

$$u(r, 0) = 0$$

$$u(r, z_B) = \int_0^r u_z(s, z_B)G(r - s, z_B)ds,$$

where $k_0 = 2\pi f / C_0$ is the reference wave number, $n(z) = C_0 / c(z)$ the index of refraction, $C_0 = 1500$ m/s the reference sound speed, $f$ the frequency of the source and $S(z, z_s)$ denotes the source term. The Neumann to Dirichlet (NtD) map boundary condition is in the form of a convolution integral and $G$ is its kernel, see [1] for more details.

## 3. THE TANGENT LINEAR MODEL

Taking the directional derivatives with respect to an unknown parameter $p$, of the WAPE boundary value problem we have

$$\frac{\partial}{\partial p}(Lu) = Lw + \frac{\partial L}{\partial p}u = 0$$

$$w(0, z) = 0$$

$$w(r, 0) = 0$$

$$w(r, z_B) = \int_0^r w_z(s, z_B)G(r - s, z_B)ds,$$

where $w = \dfrac{\partial u}{\partial p}$.

## 4. THE DERIVATIVE OF THE COST FUNCTION

We will use the following cost function (amplitude projection) proposed in [2]

$$J = \|u(R,z)\|^2 - \left( |u(R,z)|, \frac{|u_{obs}(R,z)|}{\|u_{obs}(R,z)\|} \right)^2,$$

where $(|u(R,z)|, \frac{|u_{obs}(R,z)|}{\|u_{obs}(R,z)\|}) = \int_0^{z_B} |u(R,z)| \frac{|u_{obs}(R,z)|}{\|u_{obs}(R,z)\|} dz.$

If we choose appropriate initial condition for the adjoint problem, using this inner product: $(f,g) = \int_0^R \int_0^{z_B} f(r,z)\overline{g(r,z)} dzdr,$ (see [1]), then we can obtain the derivatives of $J$ with respect to our unknown parameters, in terms of the direct field $u$ and the adjoint field $\upsilon$

$$\frac{\partial J}{\partial c_1} = \mathrm{Re}\int_0^R \int_0^{z_B} \frac{k_0 n^2}{c}(iu_r + 2k_0 u)\bar{\upsilon}(1-\frac{z}{z_m})dzdr - \mathrm{Re}\int_0^R \int_{z_m}^{z_B} \frac{k_0 n^2}{c}(iu_r + 2k_0 u)\bar{\upsilon}(1-\frac{z}{z_m})dzdr,$$

similarly for $c_2$ and $c_3$,

$$\frac{\partial J}{\partial z_m} = \mathrm{Re}\int_0^R \int_0^{z_B} \frac{k_0 n^2}{c}(iu_r + 2k_0 u)\bar{\upsilon}(c_1 - c_2)\frac{z}{z_m^2} dzdr$$

$$- \mathrm{Re}\int_0^R \int_{z_m}^{z_B} \frac{k_0 n^2}{c}(iu_r + 2k_0 u)\bar{\upsilon}[(c_3 - c_2)\frac{1}{z_B - z_m} - (c_3 - c_2)\frac{z - z_m}{(z_B - z_m)^2} + (c_2 - c_1)\frac{z}{z_m^2}]dzdr,$$

$$\frac{\partial J}{\partial a_j} = \mathrm{Re}\int_0^R \int_0^{z_B} \frac{k_0 n^2}{c}(iu_r + 2k_0 u)\bar{\upsilon} f_j dzdr, \text{ for } j = 1,2,...N.$$

## 5. NUMERICAL RESULTS

In this session we exhibit two test cases described below. A vertical array of equidistant hydrophones at a distance $R$ from the source is used, with the first and last hydrophone being $\Delta z = 1\text{m}$ far from the boundaries of the waveguide, where $\Delta z$ denotes the computational step in depth ($\Delta r$ is the step in range and is equal to 2m). IFD+IMP code is used for the numerical solution of the forward and the adjoint model and the steepest descent scheme for the minimization of the cost function (see [1] and [2]).

### 5.1 TEST CASE I

We consider a waveguide which consists of a water column with a depth of 100m and density 1.0g/cm$^3$, as well as a semi-infinite bottom with density 2.1g/cm$^3$ and sound speed 1520m/s. The source is located at 25m in depth, the array is at $R = 3\text{km}$ and we use a frequency of 50Hz. Here we consider an unperturbed SSP, see the second row of Table 1 for the actual values of the parameters $z_m$, $c_1$, $c_2$, $c_3$ and $c_B$. Note that we also seek $c_B$ (see [1] for the derivative of the cost function with respect to $c_B$).

| | $z_m$ [m] | $c_1$ [m/s] | $c_2$ [m/s] | $c_3$ [m/s] | $c_B$ [m/s] |
|---|---|---|---|---|---|
| actual values | 40 | 1500 | 1490 | 1515 | 1520 |
| initial values | 30 | 1500 | 1500 | 1500 | 1530 |

*Table 1: Actual and initial values of the parameters for the test case I.*

The initial values for the steepest descent iterative optimization scheme are shown in the third row of Table 1. In Table 2 we show the estimations obtained for $z_m$, $c_1$, $c_2$, $c_3$ and $c_B$. Note that in order to assess the efficiency of the method we show results with data which come from a Normal Mode (NM) code (MODE1 program). We observe that when we seek the water parameters $z_m$, $c_1$, $c_2$ and $c_3$, assuming that $c_B$ is known, the estimations are in good agreement with the actual values of the parameters, for 100 receivers (see the second row of Table 2). For 25 receivers, the estimation for $c_3$ is quite off (see the third row of Table 2). In the last two rows of Table 2 we show that we can recover $c_B$ together with the water parameters.

| No of receivers | $z_m$ [m] | $c_1$ [m/s] | $c_2$ [m/s] | $c_3$ [m/s] | $c_B$ [m/s] |
|---|---|---|---|---|---|
| 100 | 40.3 | 1502.2 | 1494.6 | 1512.1 | known |
| 25 | 40.1 | 1500.9 | 1490.1 | 1502.8 | known |
| 25 | 38.8 | 1504.1 | 1485.4 | 1502.3 | 1523.9 |
| 25 | known | 1503.5 | 1487.0 | 1503.1 | 1524.8 |

*Table 2: Results for the test case I.*

## 5.2 TEST CASE II

In this test case we take into account the EOFs for the SSP representation, [4] (see also [5]). The waveguide consists of a water column with a depth of 400m and density 1.0g/cm$^3$, as well as a semi-infinite bottom with density 1.5 g/cm$^3$ and sound speed 1600m/s. The source is located at 50m in depth, the array is at $R = 1\text{km}$ and we use a frequency of 150Hz. Here the background sound speed, assumed known, is 1500m/s at the surface, 1495m/s at 100m depth and 1509m/s at the bottom. We consider five different perturbations of this piecewise linear profile, see blue solid lines in Fig. 2. The actual values of the coefficients used are shown in Table 3.

| SSP | $\alpha_1$ actual | $\alpha_2$ actual | $\alpha_3$ actual |
|---|---|---|---|
| 1$^{st}$ | −19.21 | 27.845 | −11.105 |
| 2$^{nd}$ | −33.007 | 34.352 | −11.008 |
| 3$^{rd}$ | −44.713 | 44.438 | −14.892 |
| 4$^{th}$ | −25.657 | 32.824 | −13.073 |
| 5$^{th}$ | −8.726 | 22.876 | −12.001 |

*Table 3: The actual values of the coefficients.*

We choose $(0,0,0)$ as initial guess for $(a_1, a_2, a_3)$ and we also use additional constraints $(a_1 \le 0, a_2 \ge 0, a_3 \le 0)$. Note that for the IFD data the method works perfectly for 25 receivers. For the NM data, we have to point out that the estimations obtained for $a_2$ and $a_3$ are close to the actual values of the coefficients, whereas $a_1$ remains 0. In Fig. 2 we show the SSPs obtained (red dashed lines) using NM data for recovering the two ($a_2$ and $a_3$) of the three coefficients. The results are for 25 receivers. We observe that $a_1$ seems to affect the behavior of the SSPs on the top of the waveguide and we have to investigate if we can achieve an estimation for $a_1$ using more dense receivers there.

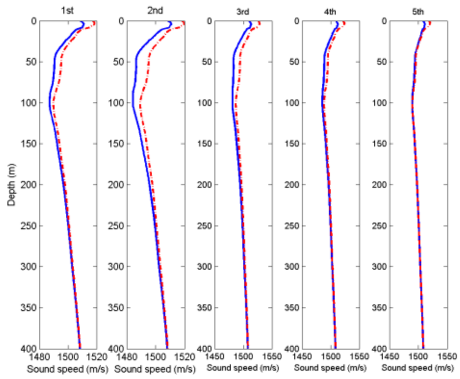


*Fig. 2: Actual (blue solid lines) and recovered (red dashed lines) for the five SSPs.*

## 6. DISCUSSION

In this report we develop analytically and numerically the adjoint optimal control method for water column SSP inversion and attempt a preliminary computational demonstration. The goal is to invert for both the background sound speed and the amplitudes of EOFs with experimental data. Therefore, we have to work in the following two directions: study of the cost function with respect to our physical parameters and appropriate choice of the descent method for minimizing the cost function.

## 7. ACKNOWLEDGEMENTS

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## REFERENCES

[1] **J. S. Papadakis**, **E. T. Flouri**, Geoacoustic Inversions based on an Adjoint Parabolic Equation with a Neumann to Dirichlet Map Boundary Condition, *Acta Acustica united with Acustica*, volume 93, pp. 924-933, 2007.

[2] **J. S. Papadakis**, **E. V. Karasmani**, Geoacoustic Adjoint-based Inversion for a layer bottom via the Parabolic Equation, In the *10th European Conference on Underwater Acoustics*, Istanbul, Turkey, Tuncay Akal, volume 1, pp. 307-314, 2010.

[3] **F. D. Tappert**, The parabolic approximation method, Wave Propagation and Underwater Acoustics, *Lecture Notes in Physics* 70, J. B. Keller and J. S. Papadakis, Springer-Verlag Heidelberg, pp. 224-287, 1977.

[4] **Taroudakis M.I. and Smaragdakis C.**, Inversions of statistical parameters of an acoustic signal in range-dependent environments with applications in ocean acoustic tomography, *J. Acoust. Soc. Am.*, volume 134, pp. 2814-2822, 2013.

[5] **Y. Jiang, N. Chapman, and P. Gerstoft,** Estimation of geoacoustic properties of marine sediment using a hybrid differential evolution inversion method, IEEE J. Oceanic Eng. volume 35, pp. 59-69, 2010.